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# BINOMIAL PAIRS, SEMI-BROUWERIAN AND BROUWERIAN SEMILATTICES

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This paper may be considered as a contribution to the axiomatization of intuitionistic logic, i.e., Brouwerian semilattices, within the wider realm of semi-Brouwerian semilattices. The latter occurred first within a purely algebraic context, as congruence lattices of semilattices (whose characterization as abstract lattices, *cf.* Grätzer [14], Problem 21, is, in a sense, still an open problem). As observed here for the first time, semi-Brouwerian semilattices form an equational class, an additional equation making them Brouwerian (Proposition 2.1). These equations have indeed a structural meaning that is fully investigated in section 1. In section 3, further conditions are given that make a semi-Brouwerian semilattice Brouwerian, some distributivity condition (Theorem 3.3) and the classical deduction theorem (Theorem 3.7) among them. Between Brouwerian and semi-Brouwerian semilattices, there is also a relationship similar to that between abelian groups and all groups (Theorem 3.6).

**1** *Binomial pairs* Let $S$ be a partially ordered set. A *closure operator* in $S$ is a mapping $\beta$: $S \to S$ such that

$$(1.1)\qquad \begin{cases} \beta(\beta(z)) = \beta(z), \\ z \leqslant \beta(z) \end{cases}$$

for each $z \in S$, moreover, for each $y, z \in S$,

$$(1.2)\qquad \textit{whenever } y \leqslant z, \textit{ then } \beta(y) \leqslant \beta(z).$$

There is a well-known one-to-one correspondence between closure operators $\beta$ and certain subsets $B \subset S$, established by

$$(1.3)\qquad B = \beta(S),\ \beta(z) = \min \{b \in B \mid z \leqslant b\}\ (z \in S).$$

The subsets $B$ occurring here are exactly those for which all those minima in (1.3) exist. The dual of a closure operator is a *kernel operator.*

A *weak closure operator* will be a mapping $\beta$: $S \to S$ just satisfying (1.1), (1.2) being no longer required. It is no longer determined by its image $\beta(S)$. E.g., in the semilattice

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