Notre Dame Journal of Formal Logic
Volume XIX, Number 1, January 1978
NDJFAM

ISOMORPHISM TYPES OF THE HYPERARITHMETIC SETS H_a

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Introduction Historically, this paper originates from M. Davis' result [1] that for $|a| = |b| < \omega^2$ $(a, b \in O)$, H_a and H_b are recursively isomorphic. Spector, in [10], showed that H_a and H_b for $|a| = |b|$ have the same Turing degree. Y. Moschovakis in [6], had shown that these results are best possible in that the sets H_a for $|a| = \alpha$, $\omega^2 \leqslant \alpha$ a principal number for addition, have well-ordered sequences of type ω_1 under one-one reducibility and, also, incomparable one-one degrees. The author in his thesis [8] showed that any countable ordered set can be embedded in the one-one ordering of H_a, $|a| = \alpha$ as above and that there are incomparable one-one degrees below any H_a, if $|a| = \alpha \geqslant \omega^3$. Moschovakis also has shown that if $\beta = \xi + \alpha$, α principle for addition, that $\{\mathsf{H}_b\colon |b| = \beta\}$ has the same structure under one-one reducibility as does $\{\mathsf{H}_a\colon |a| = \alpha\}$. This carries over to this paper after Theorem 1.1 and we restrict ourselves to those H_a such that $|a|$ is principle for addition, i.e., $|a| = \omega^\beta$ for some $\beta \geqslant 1$.

In this paper we introduce a general notion of one-one reducibility applicable to the hyperarithmetic sets (since these sets are cylinders, [9], pp. 89-90, we need only to discuss one-one reducibility). The notion is simply the following; suppose $a, b \in O$ and $|a| = |b|$, when is there a one-one function $f(x)$ recursive in H_c such that $x \in \mathsf{H}_a$ iff $f(x) \in \mathsf{H}_b$? Since H_a and H_b have the same Turing degree, clearly any $c \in O$, $|c| \geqslant |a|$ is sufficient. The question we try to answer is how small can $|c|$ be chosen in general, so that H_a and H_b are one-one reducible to each other by functions recursive in H_c, i.e., H_a and H_b are isomorphic via a permutation of $\mathcal{N}$ recursive in H_c. Alternatively, for $|c| < |a|$, H_c can be viewed as a constructive subset of both H_a and H_b and using only an oracle for H_c can one show a question of membership in H_a is equivalent to a question of membership in H_b (this is similar to a "bounded truth-table" reduction except that the bound is H_c). We will give a necessary and sufficient condition on the size of $|c|$ in order to show H_a, H_b are isomorphic by a permutation of $\mathcal{N}$ recursive in H_c when $|a| < \omega^{\omega^2}$. That this condition is sufficient for all $a, b \in O$ is demonstrated. However, the necessity of this condition for $|a| \leqslant \varepsilon_0$ is not proven and

Received September 5, 1974