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ON ACKERMANN'S THEORY OF SETS

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Two different proposals to clarify Cantor's intuitive definition of set along axiomatic lines have been made. One is the well-known theory of Zermelo and Von Neumann that a collection is a set only when it is not too large, i.e., the theory of limitation of sizes in Russell's terminology. The other is a recent system of Ackermann, *cf.* [2].

In this system; $x, y, z, \ldots$ are object variables (i.e., class variables). The primary formulae are '$x = y$', '$x \epsilon y$' (x is a member of y) and $\mathbf{M}x$ (x is a set) in which in place of x and y one can use other class variables. Further formulae or expressions can be constructed in the usual fashion with the use of logical connectives: $\neg$(not), $\wedge$(and), $\vee$(or), $\supset$ (implies), $\equiv$ (equivalent), $(\forall x)$, $(\forall y)$, . . . (for all x, for all y, . . .) and $(\exists x)$, $(\exists y)$, . . . (there is an x, there is an y, . . .). In place of $x = y$, we will use $x \neq y$, $x \subset y$ will be used as an abbreviation for $(\forall z)\,[z \epsilon x \supset z \epsilon y]$, and $\sim x \epsilon y$ will be written as $x \notin y$. We apply predicate calculus of the first degree, inclusive of calculus of equality, to these expressions (which do not contain any predicate variable). $A(x)$ is any expression which contains the free variable x; $A_0(y)$ any expression which contains the free variable y and in which the sign $\mathbf{M}$ does not occur.

The axiom system, based on Cantor's idea, contains four axiom schemata:

(α) $(\forall x)[A(x) \supset \mathbf{M}x] \supset (\exists y)(\forall z)[z \epsilon y \equiv A(z)]$.
(β) $[x \subset y \wedge y \subset x] \supset x = y$.
(γ) $[\mathbf{M}x_1 \wedge \mathbf{M}x_2 \wedge \ldots \wedge \mathbf{M}x_n] \supset [(\forall y)[A_0(y) \supset \mathbf{M}y] \supset (\exists z)$
$[\mathbf{M}z \wedge (\forall u)[u \epsilon z \equiv A_0(u)]]]$.
(δ) $[\mathbf{M}x \wedge [y \epsilon x \vee y \subset x]] \supset \mathbf{M}y$.

(γ) is thereby so comprehended that $x_1, x_2, \ldots, x_n$ are the free variables occurring in addition to y in $A_0(y)$. If these do not occur, $[\mathbf{M}x_1 \wedge \mathbf{M}x_2 \wedge \ldots \wedge \mathbf{M}x_n]$ simply drops out. It embodies the restriction that 'not every class of sets is a set'. Here (α) is for class construction (only classes of sets are sets in some cases), (β) is usual axiom of extensionality

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