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DEGREES OF UNSOLVABILITY AND STRONG FORMS OF $\mathbf{\Lambda}_R + \mathbf{\Lambda}_R \nsubseteq \mathbf{\Lambda}_R$

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1 *Introduction* In [2], J. C. E. Dekker showed that the class Λ_R of *regressive isols* is not closed under addition (equivalently, in view of [2], Proposition P 18, Λ_R is not closed under multiplication) within the ring Λ^* ([4]) of isolic integers. In [1], Barback gave a different proof of additive non-closure which, in contrast to Dekker's proof, does not appeal to the notion of *degree of unsolvability* (of a regressive isol; see [2]). Since Barback's proof makes no use of degrees, one is naturally led to wonder whether the failure of additive closure for Λ_R is *totally independent of degree*, in the sense that if $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are any given nonzero degrees of unsolvability then there exist retraceable sets α and β such that $\alpha \in \boldsymbol{d}_1$, $\beta \in \boldsymbol{d}_2$, and the respective isols A and B determined by α and β fail to have a regressive sum. In view of [5], Theorem T2 and [2], Proposition P 17, this is always the case when $\boldsymbol{d}_1 \neq \boldsymbol{d}_2$; so we need only concern ourselves with the realization of additive non-closure *within* a given degree. Barback's proof of additive non-closure can easily be embellished with enough auxiliary coding to produce one particular well-behaved class of degrees $\boldsymbol{d}$ for which the additive non-closure of Λ_R can be realized within $\boldsymbol{d}$; the class we here have in mind is $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}''\}$. As an immediate corollary to the main result of section **3**, we shall conclude that each element of the larger class $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$ bears internal witness to the additive non-closure of Λ_R ($\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$ is well known to have the interesting property of being co-extensive with the range of the jump operator ([8])); thus, we have total independence of degree for additive non-closure of Λ_R *within the class* $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$. (This much, in fact, is very easy to prove without any appeal to the limiting procedures employed in our proof of Theorem 3.1. Theorem 3.1, however, goes a bit further with respect to the form of the summands and the extent of their illrelatedness.) In section **4** we shall prove that the one-sidedness of condition (iii) in the statement of Theorem 3.1 is inescapable, at least for $n_0 = 0$. In section **5**, we restrict our attention to the subsemilattice of *recursively enumerable* degrees; there we shall establish, by means of a suitable elementary priority construction, that, for the entire class $\mathcal{J}_r - \{\boldsymbol{\phi}\}$

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