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A NOTE ON THE AXIOM OF CHOICE AND THE CONTINUUM HYPOTHESIS

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In response to a request from Roy Davies for details on the proof of Lemma 3 of [4], the author worked out his proof sketch of (5) of Lemma 1 and found it to be erroneous. In fact, if x is infinite and well-ordered, then $x^{=+} \sim x$ and so $x^{=+} \epsilon\, x^{\#}$ although $x^{=} \epsilon\, x^{=+}$ and not $x^{=} \epsilon\, x^{=}$; that is, $x^{\#} \subseteq x^{=}\mathfrak{P}$ only holds when x is finite or not well-ordered.

Although disappointing, this error is not disastrous. With the aid of an alternative to (5), an iteration of a part of the antecedent of Lemma 3 leads to Theorem 3 with an analogous iteration and all the remaining lemmas and theorems of [4]. Some additional reasoning is needed, but far less than usual in proofs of the implication of the axiom of choice by the generalized continuum hypothesis. Also, the arithmetic of transfinite numbers is not employed. It is the aim of the present note to provide the corrections and additional reasoning, but some new results are also established.[1]

In what follows, $\{\,\}$ is the empty set while $\{x\}$ and $\{xy\}$ are the sets whose only members are x and x and y respectively. Also, $x - y$ is $\{a{:}\, a \epsilon x$ and not $a \epsilon y\}$, x, y is $\{\{x\}\{xy\}\}$, and $x \times y$ is $\{a, b{:}\ a \epsilon x$ and $b \epsilon y\}$. Additional notation is as in [4]. In particular, x^{+} is $x \cup \{x\}$ and $\bigcup$ is von Neumann's operation. The set-theoretic framework employed is Zermelo-Fraenkel set theory without the axioms of choice or regularity. Since the theory of cardinal numbers cannot be developed within this framework, neither cardinal arithmetic nor cardinal notation can be employed.

In the place of (5) of Lemma 1, put

(5) $x^{\#} < x^{=}\,\mathfrak{P}\mathfrak{P}$.

There is no problem if x is finite. Assume instead that x is infinite.

1. A previous short correction notice was printed on p. 464 of the *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 17 (1971), but the author was not sent the galley proof to read. The formula "$x^{\#} \prec x^{=}\,\mathfrak{P}\mathfrak{P}$" was there misprinted as the erroneous "$x^{\#} \subseteq x^{=}\,\mathfrak{P}\mathfrak{P}$".

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