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WHEN DO *CONTINUOUS EXTENSIONS EXIST?

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In non-standard analysis we frequently need a non-standard extension of some standard function $f: X \rightarrow Y$, that is, an internal function $g: {}^*X \rightarrow {}^*Y$ other than $^*f$ such that $g|_X = f$. Often X and Y are topological spaces and we want g to be *continuous, so that for each *open $U \subseteq {}^*Y$, $g^{-1}(U) \subseteq {}^*X$ is *open. H. Gonshor showed in [3] that if X is normal, then any function $f: x \rightarrow \mathbb{R}$ has a *continuous extension. Obviously, the problem of which pairs (X, Y) have this property has some uninteresting solutions: (X, X) is such a pair where X is a discrete space. However, by introducing additional conditions on X and Y we can produce an interesting class of such pairs. First we remind the reader of the following definitions.

Definition 1 A topological space X is said to be Urysohn (or functionally Hausdorff) iff for any two points $x, y \in X$ there is a continuous function $g: X \rightarrow \mathbb{R}$ such that $g(x) = 1$ and $g(y) = 0$.

Definition 2 A topological space Y is said to be pathwise connected iff for any two points $x, y \in Y$ there is a continuous function $h: \mathrm{I} \rightarrow Y$ such that $h(1) = x$ and $h(0) = y$.

It is evident that any space Y is pathwise connected iff for any two points $x, y \in Y$ there is a continuous function $h: \mathbb{R} \rightarrow Y$ such that $h(1) = x$ and $h(0) = y$.

Theorem 1 *Let X be a Urysohn space and let Y be a pathwise connected space in a model $\mathfrak{M}$. Then for any enlargement $^*\mathfrak{M}$ of $\mathfrak{M}$, each function $f: X \rightarrow Y$ has a *continuous extension $g: {}^*X \rightarrow {}^*Y$. Moreover, if X is not a Urysohn space, then there is a function $f_1: X \rightarrow \mathbb{R}$ with no *continuous extension, and if Y is not pathwise connected, then there is a function $f_2: \mathbb{R} \rightarrow Y$ with no *continuous extension.*

Proof: Let X be a Urysohn space, let Y be a pathwise connected space and let $f: X \rightarrow Y$ be an arbitrary function. We shall show that the binary relation R defined by: $\langle x, y\rangle\, Rg$ iff $g: X \rightarrow Y$ is continuous and $g(x) = y$ is concurrent on $\{\langle x, f(x)\rangle: x \in X\}$.

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