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INDEPENDENT NECESSARY CONDITIONS FOR FUNCTIONAL COMPLETENESS IN m-VALUED LOGIC

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A function f is *functionally complete* in m-valued logic if the set of functions which can be defined explicitly from f is exactly the set of all functions of m-valued logic. A *Sheffer function* is a two-place functionally complete function. Post [1] and Webb [2], among others, have identified some Sheffer functions in m-valued logic. Martin [3] identified four properties (i.e., proper substitution, co-substitution, proper closing, and $\dagger$-closing), the absence of which are necessary conditions for functional completeness. In this paper, we will prove that co-substitution implies proper substitution; or with respect to our necessary conditions, if f does not have the proper substitution property, then f does not have the co-substitution property. Consequently, the co-substitution property can be discarded from our set of necessary conditions for functional completeness. Finally, we show that the remaining three necessary conditions are independent.

Theorem *If $f(p, q)$ is a two-place function satisfying the co-substitution property, then $f(p, q)$ satisfies the proper substitution property as well.*

Proof: Let $K = \{1, 2, 3, \ldots m\}$ be the set of m truth values, and D be a decomposition of K into $\hat{m}$ disjoint non-empty classes, $2 \leqslant \hat{m} < m$. We will say $i \sim j$ (D) if i and j are elements of the same class, $i, j \in K$. Further, let $\hat{\mathrm{D}}$ be the decomposition of the two-dimensional space K^2 such that $(p, q) \sim (r, s)(\hat{\mathrm{D}})$ if and only if $p \sim r$ (D) and $q \sim s$ (D). Let f satisfy the co-substitution law of Martin; that is, for any $h, i, j, k \in K$, whenever $f(h, i) \sim f(j, k)$ (D), then $h \sim j$ (D) or $i \sim k$ (D).

Assume there exist $(a, b) \sim (c, d)$ $(\hat{\mathrm{D}})$ such that $f(a, b) \not\sim f(c, d)$ (D). There are $\hat{m}$ - 1 classes of $\hat{\mathrm{D}}$, we will call them $C_1, C_2, \ldots C_{\hat{m}-1}$, such that if $(w_i, x_i) \in C_i$ then $a \not\sim w_i$ (D), $b \not\sim x_i$ (D), $c \not\sim w_i$ (D), and $d \not\sim x_i$ (D). Further, if $(w_i, x_i) \in C_i$ and $(w_j, x_j) \in C_j$, $i \neq j$, then $w_i \not\sim w_j$ (D) and $x_i \not\sim x_j$ (D). Since f satisfies co-substitution, $f(a, b) \not\sim f(w_1, x_1)$ (D) and $f(c, d) \not\sim f(w_1, x_1)$ (D). Further, $f(a, b) \not\sim f(w_2, x_2)$ (D), $f(c, d) \not\sim f(w_2, x_2)$ (D) and $f(w_1, x_1) \sim f(w_2, x_2)$ (D). Continuing, we reach the case that $f(w_{\hat{m}-1}, x_{\hat{m}-1})$

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