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INCOMPLETE TRANSLATIONS OF COMPLETE LOGICS

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Let J and K be sets of (interpreted) logical primitives and let **LJ** and **LK** be languages based on J and K respectively, but having a common set of variables and non-logical constants. Let $\mathcal{L}J$ be a logic on **LJ**. Suppose **t** is a function which carries formulas of **LJ** into logically equivalent formulas of **LK**. It has been known since at least 1958 [6] that the completeness of the logic on **LK** ($\mathcal{L}K$), resulting from the translation (by **t**) of $\mathcal{L}J$ is not assured by the completeness of $\mathcal{L}J$.

This result may not be widely known; in 1972 Crossley [2] made a mistake by overlooking it. Crossley constructed a logic, here called $\mathcal{L}[\neg, \&, \exists]$, by translating a logic known to be complete,[1] here called $\mathcal{L}[\cdot, \rightarrow, \forall]$. Crossley thought that $\mathcal{L}[\neg, \&, \exists]$ is complete, but it is not.[2] Similar examples may have motivated William Frank's recent article [3] in this *Journal* concerning the reasons why some translations do not preserve completeness. Unfortunately, there are two errors in the latter; it is the purpose of this article to set them straight. Frank's main theorem reads as follows:

> *If* T(A) *is the closure of a formal system in a language* $\mathcal{L}$, *with axioms* A1, . . ., AN; *and rules* R1, . . ., RM *and* **t** *a rule of translation from* $\mathcal{L}$ *to* $\mathcal{L}'$, *then* T$'$, *the closure of* **t**(A1), . . ., **t**(AN), **t**(R1), . . ., **t**(RM), *is equal to* **t**(T(A)).

In other words, the only theorems in $\mathcal{L}'$ are translations of theorems in $\mathcal{L}$.

Let $\mathcal{L}$ have 3 sentences: a, b, and c; one axiom: a; and one rule: b/c; so only one theorem: a. Let $\mathcal{L}'$ have two sentences: A, B. Let $\mathbf{t}(a) = A$, $\mathbf{t}(b) = A$, $\mathbf{t}(c) = B$. $\mathcal{L}'$ will then have two theorems: A, B because $\mathbf{t}(a) = A$ is an axiom and $\mathbf{t}(b)/\mathbf{t}(c) = A/B$ is a rule. But B is not the translation of a theorem in $\mathcal{L}$. The problem is that the translation of a non-rule (a/b) can become a rule if the translation is not 1-1.

1. Typographical errors in axiom 5 of [2], p. 19, are assumed to be corrected.

2. For example, some instances of $A \;\&\; A \rightarrow A$ are not provable (see below).

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