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NEXT $\mathcal{P}$ ADMISSIBLE SETS ARE OF COFINALITY ω

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The first and most direct generalization of the Barwise compactness theorem to the uncountable case was the cofinality ω compactness theorem of Barwise and Karp [1], [2]—a power set admissible set which can be written as a union of countably many of its elements is Σ_1 (in the graph of the power set) compact. Thus, in order to directly generalize the many situations in which the Barwise compactness theorem is applied to a next admissible set, we need to know that all next power set admissible sets can be written as appropriate countable unions. In this paper* we show, using elementary methods, that they can. A modification of the proof of Theorem 5.3 of [1] can also be used but involves higher order predicates.

We assume familiarity with the notion of power set admissibility, presented for example in [2], and the fact that any power set admissible set can be written as a $V(\kappa)$. We also will use the obvious fact that there are only countably many formulas which are Δ_0 in the graph of the power set and abuse notation slightly by calling these Δ_0 in $\mathcal{P}$ formulas. For each cardinal λ we let $\beth_0(\lambda) = \lambda$, $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$, and $\beth_\omega(\lambda) = \bigcup_{n \in \omega} \beth_n(\lambda)$.

Theorem *Every next power set admissible set is of cofinality ω.*

Proof: Suppose $V(\kappa)$ is the smallest power set admissible set containing the set A and $\kappa_0 = \beth_\omega(\rho)$ where ρ is the cardinality of the rank of A. Clearly $A \in V(\kappa)$ and $V(\kappa)$ $\mathcal{P}$ admissible implies $\kappa_0 < \kappa$. Starting with κ_0 we construct a sequence of cardinals, each of cofinality ω, such that for each n, $\kappa_{n-1} < \kappa_n \leqslant \kappa$. If at any time we find $\kappa_n = \kappa$ we are done so we may assume $\kappa_0 < \kappa_1 < \ldots < \kappa_n < \kappa$ and for $j \leqslant n$, $\kappa_j = \bigcup_{m \in \omega} \kappa_{j,m}$. Since we will eventually want to show that $V\left(\bigcup \kappa_n\right)$ is $\mathcal{P}$ admissible $\left(\text{and hence } \kappa = \bigcup \kappa_n\right)$ we want to construct the sequence to satisfy:

if Q is any $k+2$ place Δ_0 in $\mathcal{P}$ formula and $a, b_1, \ldots, b_k \in V\left(\bigcup \kappa_n\right)$ then $\forall x \in a \exists y \in V\left(\bigcup \kappa_n\right) Q(x, y, b_1, \ldots, b_k)$ implies there is an $n \in \omega$ such that $\forall x \in a \exists y \in V(\kappa_n)\, Q(x, y, b_1, \ldots, b_k)$.

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