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NEGATION, MATERIAL EQUIVALENCE, AND CONDITIONED NONCONJUNCTION: COMPLETENESS AND DUALITY

GERALD J. MASSEY

1 *Object of paper* The object of this paper is threefold: to prove the functional incompleteness of $\{\sim, \equiv\}$ without appeal to a tedious analysis of cases; to give a proof simpler than the one in [2], p. 284 f., of the nonexistence of indigenous Sheffer connectives for $\{\sim, \equiv\}$; and to furnish self-dual (in Church's sense) ternary Sheffer connectives for propositional logic.

2 *Functional incompleteness of* $\{\sim, \equiv\}$

Lemma 1 *Let A be a propositional wff containing no connectives other than $\sim$ and $\equiv$, and let A' be the wff that results when all occurrences of $\sim$ in A are deleted. Then A is equivalent to A' or to $\sim A'$.*

Proof: Let $\Leftrightarrow$ signify (semantic) equivalence. Since $B \Leftrightarrow \sim\sim B$ and since $\sim(B \equiv C) \Leftrightarrow \sim B \equiv C \Leftrightarrow B \equiv \sim C$, Lemma 1 follows from the substitutivity of equivalents by induction.

Theorem 1 $\{\sim, \equiv\}$ *is functionally incomplete.*

Proof: We call a truth-value α a *fixed point for* a wff B just in case the value of B is α when all its variables are assigned the value α, and we say that α is a *fixed point for* an n-ary connective $\otimes$ just in case α is a fixed point for $\otimes(p_1, \ldots, p_n)$. Notice that $\mathbf{t}$ is a fixed point for $\equiv$ and for any wff that contains no connectives other than $\equiv$. Suppose that & is definable from $\{\sim, \equiv\}$. Then by Lemma 1 there is a wff $A(p, q)$ containing no connectives other than $\equiv$ such that $p \,\&\, q$ is equivalent to $A(p, q)$ or to $\sim A(p, q)$. But since $\mathbf{t}$ is a fixed point for both $p \,\&\, q$ and $A(p, q)$, $p \,\&\, q$ cannot be equivalent to $\sim A(p, q)$. So, $p \,\&\, q \Leftrightarrow A(p, q)$. Therefore, $p \,\&\, \sim q \Leftrightarrow A(p, \sim q)$. By Lemma 1 again, $A(p, \sim q)$ is equivalent to $A(p, q)$ or to $\sim A(p, q)$. Since $\mathbf{t}$ is not a fixed point for $p \,\&\, \sim q$, we have $A(p, \sim q) \Leftrightarrow p \,\&\, \sim q \Leftrightarrow \sim A(p, q)$. So, $\sim(p \,\&\, \sim q) \Leftrightarrow A(p, q) \Leftrightarrow p \,\&\, q$, which is a contradiction. So & is not definable from $\{\sim, \equiv\}$. Theorem 1 follows.

3 *Indigenous Sheffer connectives* An n-ary connective $\otimes$ is said to be a

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