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A NOTE ON EVALUATION MAPPINGS

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Let $\mathcal{L}$ be a functionally complete sentential language. Let $\Phi: \mathcal{L}^n \times \mathcal{A} \to \{0, 1\}$, where $n \geq 1$ and $\mathcal{A}$ is the set of all assignments (i.e., mappings from the set V of all variables to $\{0, 1\}$). Then Φ shall be called an *evaluation mapping on* $\mathcal{L}$ in case for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$ and all $\mathfrak{A}, \mathfrak{A}' \in \mathcal{A}$, if $\mathfrak{A}$ and $\mathfrak{A}'$ agree on the variables occurring in $\varphi_1, \ldots, \varphi_n$ then $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = \Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}')$. The notion of evaluation mapping is a syntactico-semantic generalization of the usual notion of truth-functional connective. For $S \subseteq \mathcal{L}$ and Φ an (n-ary) evaluation mapping:

(1) Φ is *truth-functional on* S in case for all $\varphi_1, \ldots, \varphi_n, \varphi_1', \ldots, \varphi_n' \in S$ and $\mathfrak{A}, \mathfrak{A}' \in \mathcal{A}$, if $V_{\mathfrak{A}}(\varphi_i) = V_{\mathfrak{A}'}(\varphi_i')(1 \leq i \leq n)$, then $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = \Phi(\varphi_1', \ldots, \varphi_n', \mathfrak{A}')$.

(2) Φ is *Boolean on* S in case there is $\varphi \in \mathcal{L}$ with n variables such that for all $\varphi_1, \ldots, \varphi_n \in S$ and every $\mathfrak{A} \in \mathcal{A}$, $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = V_{\mathfrak{A}}\left(\varphi\begin{bmatrix}\alpha_1, \ldots, \alpha_n\\ \varphi_1, \ldots, \varphi_n\end{bmatrix}\right)$, where $\alpha_1, \ldots, \alpha_n$ are the variables occurring in φ, $\varphi\begin{bmatrix}\alpha_1, \ldots, \alpha_n\\ \varphi_1, \ldots, \varphi_n\end{bmatrix}$ is the sentence resulting from the simultaneous substitution in φ of φ_i for α_i $(1 \leq i \leq n)$, and $V_{\mathfrak{A}}$ is the sentential valuation induced by $\mathfrak{A}$.

Theorem *For every* $S \subseteq \mathcal{L}$ *and every evaluation mapping* Φ, Φ *is Boolean on* S *if and only if* Φ *is truth-functional on* S.

Proof: Necessity is obvious. We prove sufficiency. Suppose that $\Phi: \mathcal{L}^n \times \mathcal{A} \to \{0, 1\}$ is truth-functional on S. Let $f: \{0, 1\}^n \to \{0, 1\}$ be the Boolean function such that for all $x_1, \ldots, x_n \in \{0, 1\}, f(x_1, \ldots, x_n) = \Phi(p_1, \ldots, p_n, \mathfrak{A})$, where $\mathfrak{A}(p_i) = x_i$ $(1 \leq i \leq n)$, and $p_1, \ldots, p_n$ are the first n variables of V. Then, by the definition of evaluation mapping (the full force of truth-functionality not being needed), f is well-defined, independent of the choice of $\mathfrak{A}$. Let $\varphi(= \varphi(p_1, \ldots, p_n)) \in \mathcal{L}$ express the function f. Then, for every $\mathfrak{A} \in \mathcal{A}$ and for all $\varphi_1, \ldots, \varphi_n \in S$, letting $\mathfrak{A}' \in \mathcal{A}$ such that $\mathfrak{A}'(p_i) = V_{\mathfrak{A}}(\varphi_i)(1 \leq i \leq n)$, we have (since Φ is truth-functional) that

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