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ON THE INDEPENDENCE OF THE FUNDAMENTAL OPERATIONS OF THE ALGEBRA OF SPECIES

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The aim of this article is to prove the results announced in [2]. That is, that the fundamental operations of the algebra of species are independent, in the sense that none of the four operations is definable in terms of the others. The fundamental operations of the algebra of species are: the species implication "$\Rightarrow$", the species union "$\cup$", the species intersection "$\cap$", and the species complement "$-$". For details see [6]. A species algebraic operation, say "$\Rightarrow$", is definable in terms of the other operations, if given any term T of the algebra of species which contains "$\Rightarrow$" and does not contain any of the other three operations, there exists a term T^* which does not contain "$\Rightarrow$", such that the formula $(T \Rightarrow T^* \cap T^* \Rightarrow T) = 1$ is valid in every algebra of species. We shall call this the defining formula of "$\Rightarrow$". We shall in each case prove the independence of each operation by giving a species algebra in which the defining formula of the operation is not valid.

Definition 1 Let $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$ be an algebra of species, and T a term of the algebra of species. The following formulae define (recursively) a function $F_{T,\mathfrak{A}}$ which correlates an element $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \in S$ with every infinite sequence of elements $X_1, \ldots, X_n, \ldots \in S$:

(i) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = X_p$ if $T = T_p (p = 1, 2, 3 \ldots)$;
(ii) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \Rightarrow F_{T_2,\mathfrak{A}}(X_1, \ldots, X_n, \ldots)$, if $T = T_1 \Rightarrow T_2$ (where T_1 and T_2 are terms);
(iii) and (iv) Analogously for the operations "$\cup$" and "$\cap$";
(v) $F_{T,\mathfrak{A}}\overline{(X_1, \ldots, X_n, \ldots)} = F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n)$, if $\overline{T} = T_1$.

We say that a term T is verified by the species algebra, in symbol $T \in \mathrm{E}(\mathfrak{A})$, if $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = A$ for all $X_1 \ldots X_n \in S$.

Definition 2 A formula $\mathfrak{H} = (T = 1)$ of the algebra of species is said to be valid in the algebra $\mathfrak{A}$, if T is satisfied by $\mathfrak{A}$, and $\mathfrak{H}$ is said to be valid in the algebra of species if it is valid in every algebra of species.

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