Notre Dame Journal of Formal Logic
Volume XVII, Number 3, July 1976
NDJFAM

A MODEL-THEORETIC SEMANTICS FOR MODAL LOGIC

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This paper will deal with the semantics of modal logic in a model-theoretic way. This semantics is clearer than the standard accounts and furthermore will lend itself to further development. From it will flow all the completeness results of modal logic along with a few new results. The paper will presuppose some familiarity with modal logic and model theory.

The key to this semantics lies in the notion of a many-sorted (two-sorted for simplicity) predicate logic. A many-sorted language is simply a language with symbols for sorts and symbols for relations and constants. A structure $\mathfrak{M}$ of similarity type τ ($\mathfrak{M} \in \mathsf{Str}(\tau)$) will associate to each sort symbol of τ a universe of objects. The universe of $\mathfrak{M}$ will be the union of the universes of each sort. $\mathfrak{M}$ will associate to each n-ary relation symbol of τ an n-ary relation over the universe and to each constant symbol of τ an individual in the universe. Satisfaction of many-sorted logic is defined just as in one-sorted logic.

A vector space is a typical example of a structure of such a language. Here there are two different sorts of entities: scalars and vectors. There are also relations on the vectors, relations on the scalars, and relations on both.

Theorem 1 *Let* L *be a predicate logic of type* τ. *Then there is a type* τ^{Σ} *having at least one new sort* I *such that we have three operations satisfying four conditions. The three operations are:*

(i) *from the* L*-sentences of type* τ *to the* L*-sentences of type* τ^{Σ}, $\phi \rightarrow \phi^{\Sigma}$;
(ii) *from a set of structures of type* τ *to a structure of type* τ^{Σ}, $\langle \mathfrak{M}_i \rangle_{i \in \mathsf{I}} \rightarrow \mathfrak{M}^{\Sigma}$;

and

(iii) *from a* τ^{Σ} *structure to a* τ*-structure*, $\mathfrak{M}^{\Sigma} \rightarrow \mathfrak{M}_i$.

The conditions these operations satisfy are as follows:

(i) $\mathfrak{M}^{\Sigma} \in \mathsf{Str}(\tau^{\Sigma})$ *whenever* $\mathfrak{M}_i \in \mathsf{Str}(\tau)$ *for each* $i \in \mathsf{I}$ *and* $\mathfrak{M}^{\Sigma} = (\mathsf{I}, \ldots)$;

Received December 13, 1974