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# AN ARITHMETICAL RECONSTRUCTION OF THE LIAR'S ANTINOMY USING ADDITION AND MULTIPLICATION

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The present note gives an improvement on [8]. There it was shown that the liar's antinomy can be reconstructed in any recursively enumerable arithmetical theory in which all elementary functions are definable, if the theory is assumed to be complete.[1] Here the same construction is done by requiring only definability of addition and multiplication. This constitutes a natural and therefore straight-forward proof of a strong version of the incompleteness theorem for arithmetical theories. The improvement on [8] consists in the fact that addition and multiplication are obviously fewer and less complex than all elementary functions: the former belong respectively to the first and to the second class of Grzegorczyk's hierarchy [5], whereas the latter constitute its third class. The present result can be considered optimal in so far as it is impossible to obtain the same result for the less complex function used because definability of addition alone allows completeness, see [1].

**1** *Nomenclature* A traditional first-order arithmetical language is usually constructed from the following items: individual variables $x_0, x_1, \ldots$; an individual constant 0 to represent the number zero; a unary function symbol $s$ to represent the successor function; a finite (possibly empty) set of binary operation symbols $\{\circ_i | 1 \leq i \leq a\}$, where $a$ is any natural number, to represent e.g., addition, multiplication, etc.; the equality symbol $=$; connectives, say $\neg$ and $\rightarrow$; quantifiers, say $\wedge$ and $\vee$.

We consider any such language $\mathcal{L}$. Let $\mathbf{N}$ be the set of natural numbers and $n, n_1, n_2, \ldots$ be any elements of $\mathbf{N}$. Let $\theta, \theta_1, \theta_2$ be any terms of $\mathcal{L}$ and let $\Phi, \Phi_1, \Phi_2$ be any formulas of $\mathcal{L}$. We have to define a computable injection $g$: $\mathcal{L} \rightarrow \mathbf{N}$. We set

$$b = a + 8$$
$$\langle n_1, n_2 \rangle = (n_1 + n_2)^2 + n_1$$

1. Note in [8] the following misprints: p. 377, l. 23: $\varepsilon^2$ *instead of* $\varepsilon^3$; p. 378, l. 10: quantifier *instead of* quantifier $\vee$; l. 17: $i \in n$ *instead of* $i \leq n$; l. 30: negative *instead of* nonnegative; p. 379, ll. 15, 17, 19, 22, 23, 24, 26, 27: $\forall$ *instead of* $\vee$; p. 380, l. 2: sur *instead of* zur.

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