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ON NACHBIN'S CHARACTERIZATION OF A BOOLEAN LATTICE

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A classical theorem of L. Nachbin [6] characterizes Boolean lattices as those bounded distributive lattices in which each prime ideal is maximal. This result has been generalized and applied to non-bounded distributive lattices by G. Grätzer and E. T. Schmidt, see [3], especially p. 276. Recently, D. Adams ([1], Theorem 1) has given a version of Nachbin's theorem for bounded non-distributive lattices. The object of this note is to give a transparent alternative proof of Grätzer and Schmidt's generalization and also to establish a theorem akin to that of Adams.

The notation and terminology follows that of [2] and Stone's Theorem ([2], Theorem 15, p. 74) will be used freely. Incidentally, a proof of Nachbin's Theorem is given in [2], Theorem 22, p. 76; it is a simplication (possibly due to boundedness) of the proof in [3]. For elements x and y of a lattice $\mathfrak{L}$, let $\langle x,y\rangle = \{z \in L\colon\ x \wedge z \leqslant y\}$. When L is distributive, $\langle x,y\rangle$ is an ideal. For a detailed account of such ideals, see Mandelker [5].

The following lemma is an extension of [4], Lemma 12.

Lemma 1 *A distributive lattice* $\mathfrak{L}$ *is relatively complemented if and only if for each* $x,\ y \in L$, $(x] \vee \langle x,y\rangle = L$.

Proof: Suppose $\mathfrak{L}$ is relatively complemented and x, y, z are in L. Let w be the complement of x in $[x \wedge y \wedge z,\ x \vee y \vee z]$. Then, $z = z \wedge (x \vee y \vee z) = z \wedge (x \vee w) = (z \wedge x) \vee (z \wedge w)$. Since $z \wedge x \in (x]$ and $z \wedge w \in \langle x, y\rangle$, it follows that $(x] \vee \langle x, y\rangle = L$.

Conversely, suppose the ideal-theoretic condition holds. Let $c \in [a, b]$. Then, $b \in (c] \vee \langle c, a\rangle$ and so $b = c_1 \vee d$ for some $c_1 \leqslant c$ and $d \in L$ such that $c \wedge d \leqslant a$. Then $b = c \vee d$ and $(d \vee a) \wedge b$ is the relative complement of c.

Lemma 2 *The set of prime ideals of a distributive lattice* $\mathfrak{L}$ *is unordered by set-inclusion if and only if, for each* $x,\ y \in L$, $(x] \vee \langle x, y\rangle = L$.

Proof: Suppose the set of prime ideals is unordered. If $(x] \vee \langle x, y\rangle \neq L$ then there is a prime ideal P such that $(x] \vee \langle x, y\rangle \subseteq P$. Since the set of prime filters is unordered, $L\backslash P$ is a maximal filter. But $x \notin L\backslash P$. Hence,

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