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Material Implication in Orthomodular (and Boolean) Lattices

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1 Introduction A number of mathematical structures have been investigated in the quantum logic approach to the foundations of quantum mechanics (see, e.g., [21]), but orthomodular lattices (OMLs) have received the majority of attention from logicians.* When interpreting such structures from the viewpoint of logic, it is customary to interpret the lattice elements as propositions, the meet operation as conjunction, the join operation as disjunction, and the orthocomplement operation as negation. It is also customary to interpret the partial order relation as the *relation* of implication.

In classical logic, if we regard propositions as sets of possible worlds, then we have the following analogs: The set of all subsets of the set W of possible worlds is a Boolean (ortho)lattice (BL), where meet (conjunction) is set-intersection, join (disjunction) is set-union, and orthocomplementation (negation) is set-complementation. On the other hand, the partial-order relation is the set-inclusion relation, which represents the *relation* of implication among propositions.

Implication in this sense is quite different from the logical connectives. For example, whereas the conjunction (intersection) of two propositions A and B is another proposition just like A and B, the inclusion of A in B is not; there is no set of worlds corresponding to "A implies B", in *this* sense of 'implies'.

There is an analogy in formal logic. Recall that a formula ϕ *semantically entails* a formula ψ just in case every interpretation that satisfies ϕ also satisfies ψ. If we regard an interpretation as assigning a proposition to each formula, then we can equivalently state this as follows: ϕ semantically entails ψ just in case for every interpretation i, $i(\phi)$ implies $i(\psi)$, or more concretely, $i(\phi)$ is

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