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Forking in Modules

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In this paper I am going to establish a connection between algebraic properties of modules and forking for types over theories of modules. Forking was invented by Shelah [13]. The notion "does not fork" is a generalization of natural notions of independence such as algebraic independence in fields and linear independence in vector spaces. In modules it is reasonable to consider two nonzero elements to be independent if they lie in complementary direct summands.

I would like to prove that if $\mathcal{A}$ is a module, $a \in \mathcal{A}$, and $B \subset \mathcal{A}$ then $tp(a; B)$ does not fork over ϕ if and only if a and B are contained in complementary direct summands of $\mathcal{A}$. Unfortunately, this statement is false. There are two things which go wrong. First, the module $\mathcal{A}$ may not have many direct summands. This can be remedied by considering summands of $\overline{\mathcal{A}}$, the pure-injective envelope of $\mathcal{A}$, which is an elementary extension of $\mathcal{A}$. The second problem is that algebraic elements are not distinguished by forking. For example, suppose $\mathcal{A} \simeq Z(p^\infty)$, $\mathcal{B} \simeq Z(p^\infty)$, $a \in \mathcal{A}$, $c \in \mathcal{A}$, $b \in \mathcal{B}$, $a \neq 0$, $c \neq 0$, $b \neq 0$, and $pa = pc = pb = 0$. Then in $\mathcal{A} \oplus \mathcal{B}$, $(a, 0)$ and $(0, b)$ should clearly be considered independent whereas $(a, 0)$ and $(c, 0)$ should not, but nevertheless neither $tp((c, 0); (a, 0))$ nor $tp((0, b); (a, 0))$ forks over ϕ.

To avoid the complications arising from the presence of algebraic elements, I will embed the original module in a pure extension in which there are no nonzero algebraic elements, and then I will consider forking in that extension. Let $\mathcal{A}$ be a module. $\mathcal{A}^\omega$ is the direct product of ω copies of $\mathcal{A}$, and Δ: $\mathcal{A} \rightarrow \mathcal{A}^\omega$ is the diagonal embedding defined by $\Delta(a) = (a, a, a, \ldots)$. It is easy to see that Δ is a pure embedding. Then the theorem I will prove is that if $a \in \mathcal{A}$ and $B \subset \mathcal{A}$ then a and B lie in complementary direct summands of $\overline{\mathcal{A}}$ if and only if $tp(\Delta(a); \Delta(B))$ does not fork over ϕ.

Several different presentations of forking have been given ([1], [9], [13]). I will follow Baldwin's version, which is based on Lascar and Poizat's treatment.

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