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A Highly Efficient "Transfinite Recursive Definitions" Axiom for Set Theory

ROBERT S. WOLF

Introduction We will consider formal set theories with an axiom (schema) that is well-known as a theorem or principle of set theory, but to our knowledge has not been proposed previously as an axiom. Essentially, this new schema *RD* says that the epsilon relation is one on which functions can be defined by transfinite recursion. At first, one might suspect that this would be equivalent to the more familiar axiom schema of (transfinite) induction on epsilon. With enough other axioms present, this is true. At the same time, we find (as is sometimes the case) that recursion is in some ways a more powerful principle, and that this effect becomes more pronounced when the underlying logic is intuitionistic rather than classical.

Specifically, we will see that the technical situation is as follows:

1. Classically, even our strongest version of *RD* (out of three versions) is provable in *ZFC* set theory. So *RD* gives nothing new. Its main feature in this context is its great "efficiency": when added to just a few of the axioms of *ZF*, *RD* suffices to derive the full axioms of replacement, choice, and foundation in any form (including induction on epsilon). Thus *RD* can be used to give a very "short" axiomatization of *ZFC*.

2. Our use of *RD* arose while studying axiomatic set theories with the law of the excluded middle confined to bounded predicates–theories which we have called "partially intuitionistic". In this context, we find that the efficiency and strength of *RD* are even more striking. It is still the case that *RD*, plus a few basic axioms, proves replacement in any reasonable form, choice, and induction on epsilon (though certain other forms of foundation don't seem to follow). In the other direction, the two weaker forms of *RD* are still derivable from more standard axioms. However, full *RD* is not provable in any straightforward way even from very strong forms of these standard axioms. We

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