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On Fleissner's Diamond

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Fleissner [1], in the course of showing that $V = L$ implies every normal topological space is collectionwise Hausdorff, used a strengthening of Jensen's $\diamond$ principle, denoted $\diamond_{SS}$, and often called "diamond for stationary systems". Mathias [3] stated $\diamond_{SS}$ explicitly and asked whether for $\aleph_1$, for example, $\diamond_{SS}$ follows from the related principles $\diamond^*_{\aleph_1}$ or $\diamond^+_{\aleph_1}$. The purpose of this paper* is to show that these implications may fail even under relatively nice conditions. This result was announced in [4].

For the remainder of the paper λ denotes a regular uncountable cardinal and S a stationary subset of λ. The reader may, for simplicity, want to identify λ with $\aleph_1$.

We now introduce the various sorts of $\diamond$-sequences under consideration and mention some of the connections between them.

Definition 1 A sequence $\langle A_\alpha: \alpha \in S\rangle$ is a *$\diamond_S$-sequence* if for each $\alpha \in S$, $A_\alpha \subseteq \alpha$ and for every $A \subseteq \lambda$, $\{\alpha \in S: A \cap \alpha = A_\alpha\}$ is stationary (in λ).

Definition 2 A sequence $\langle P_\alpha: \alpha \in S\rangle$ is a *weak $\diamond_S$ sequence* (w-$\diamond_S$ sequence) if each P_α is a set of subsets of α, and for every $A \subseteq \lambda$, $\{\alpha: A \cap \alpha \in P_\alpha\}$ is stationary. If, in addition, $\overline{\overline{P_\alpha}} \leqslant \overline{\overline{\alpha}}$ for each $\alpha \in S$, we call $\langle P_\alpha: \alpha \in S\rangle$ a *$\diamond_S$-sequence.*

The above definitions obviously involve an abuse of terminology. Notice however, that $\langle A_\alpha: \alpha \in S\rangle$ is a $\diamond_S$-sequence in the sense of Definition 1 iff $\langle \{A_\alpha\}: \alpha \in S\rangle$ is a $\diamond_S$-sequence in the sense of Definition 2.

Kunen has proved the following result relating the existence of the two types of $\diamond_S$-sequences.

Theorem 1 (Kunen) *If there is a $\diamond_S$-sequence $\langle P_\alpha: \alpha \in S\rangle$, with $P_\alpha \subseteq P(\alpha)$, then there is a $\diamond_S$-sequence $\langle A_\alpha: \alpha \in S\rangle$, with $A_\alpha \subseteq \alpha$.*

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