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MODELS OF AN EXTENSION OF THE THEORY ORD

NICHOLAS J. de LILLO

In [1], the first-order theory **ORD** was introduced as a concrete example of a theory in which the proof-theoretic concepts of implicit and explicit definability can be illustrated. Here, we use the concept of implicit definability as a means of constructing a conservative extension of certain first-order theories. The construction is then applied to **ORD** to yield a conservative extension **ORD***. It is then shown that, under certain closure conditions on the domain A of any of the underlying models of **ORD***, A is (up to isomorphism) an ordinal. Thus, in this sense, **ORD*** is a formal characterization of ordinal numbers in first-order logic.

1 *The axioms of* **ORD** *and* **ORD*** As was described in [1], **ORD** is a first-order theory with equality, with the four binary relation symbols $\approx$, $\subset$, $\subseteq$, and ϵ representing the only extra-logical symbols in its alphabet. The axiom set Γ_0 for **ORD** consists of the universal closures of the following ten wffs:

(O_1) $[(x \subset y) \wedge (y \subset z)] \rightarrow (x \subset z)$
(O_2) $\sim (x \subset x)$
(O_3) $(x \subset y) \rightarrow \sim (y \subset x)$
(O_4) $(x \subset y) \vee (y \subset x) \vee (x \approx y)$
(O_5) $(x \subseteq y) \leftrightarrow \{[(\forall z)[(z \subset x) \rightarrow (z \subset y)] \wedge \sim (x \approx y)] \vee (x \approx y)\}$
(O_6) $[(x \approx y) \wedge (z \approx u)] \rightarrow [(x \subset z) \rightarrow (y \subset u)]$
(O_7) $[(x \approx y) \wedge (z \approx u)] \rightarrow [(x \subseteq z) \rightarrow (y \subseteq u)]$
(O_8) $[(x \approx y) \wedge (z \approx u)] \rightarrow [(x \approx z) \rightarrow (y \approx u)]$
(O_9) $(x \subseteq x)$
(O_{10}) $(x \approx x)$.

Let Γ_0' be the set of six sentences (O_1')-(O_6') obtained from Γ_0 by systematically replacing each occurrence of the symbol $\subset$ in (O_1) through (O_6) by an occurrence of the symbol ϵ. Thus, for instance, (O_1') is the universal closure of the wff $[(x \epsilon y) \wedge (y \epsilon z)] \rightarrow (x \epsilon z)$. Further, let **ORD*** be the theory whose non-logical axioms are $\Gamma_0 \cup \Gamma_0'$. Clearly, **ORD*** is a first-order extension of **ORD**.

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