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AXIOMATICS FOR IMPLICATION

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This paper presents a basic axiomatic and two increments thereto, for propositional systems with implication as the sole functor. The basic axiomatic gives exactly the set of Modus Ponens formulae (defined below); addition of the first increment gives Positive Logic; and with addition of the second increment we reach the complete Classical Logic.
After some preliminaries in section 1, the axiomatics are presented in section 2. Section 3 establishes their properties.

1 *Preliminaries. Modus ponens formulae* Lower case Greek letters, with and without subscripts, are used for well-formed propositional formulae whose only functor is implication. Braces—'{' and '}'—form ordered sets of such formulae. '$\sim$' denotes a relationship between an ordered set of formulae and a single formula which is defined below.

Definition 1 $\{\alpha_1, \ldots, \alpha_n\}$ *closes* β (written $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$) is defined inductively in two steps.

I. Let there be some α_i $(1 \leqslant i \leqslant n)$ such that $\alpha_i = \beta$: then $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

II. Let there be some γ such that $\{\alpha_1, \ldots, \alpha_n\} \sim C\gamma\beta$ and $\{\alpha_1, \ldots, \alpha_n\} \sim \gamma$: then $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

Definition 2 $C\alpha_1 \ldots C\alpha_n\beta$ $(n \geqslant 1)$ is a *Modus Ponens formula* iff β is elementary and $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

Examples of Modus Ponens formulae are: *Cpp*, *CpCqp*, *CCpCqrCCpqCpr*. Formulae which are not Modus Ponens formulae are: *CCCpqrCqr*, *CCCprsCCCqprs*.

2 *Axiomatics* The three axiomatic systems are based on a single axiom, and—including substitution—six inference rules. Axiom and rules are as follows.

Axiom. *Cpp*

Rule 1. Where $[x/\beta]\alpha$ is the result of replacing every occurrence of the variable x in α by β. $\alpha \vdash [x/\beta]\alpha$

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