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Independence in Higher-Order Subclassical Logic

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Introduction A formal logic can be distinguished in a variety of ways. In the case of intuitionistic logic or classical logic it may carry an intended interpretation. If that is lacking, however, it still can be distinguished by the theorems it generates. Indeed, between the intuitionistic and the classical there extends a whole hierarchy of uninterpreted logics (the so-called intermediate logics), generally known by their theorems alone.

A third point of view is possible. Within the logic certain interdefinabilities of connectives and quantifiers may arise and this would allow one to study the formalism for redundancy (or lack of it) among the primitives used to present it. First-order classical logic has a high degree of redundancy, typical examples being the dualities between $\vee$ and & and between $\exists$ and $\forall$. In higher-order classical logic we have the curious fact (reported by Henkin [2]) that all connectives and quantifiers are definable just from equality. Intuitionistic logic presents a quite different picture. Prawitz [3] reports a complete independence among connectives and quantifiers within first-order intuitionistic logic. Passing to higher-order intuitionistic logic, he finds a return to redundancy involving a distinguished pair of primitives. He shows that in higher-order intuitionistic logic all connectives and quantifiers are definable from $\rightarrow$ and $\forall$ alone.

A central purpose of this paper is to reexamine the $\rightarrow$, $\forall$ definabilities of Prawitz from the point of view of independence. The question to ask is obviously this: Are there other redundancies possible for higher-order intuitionistic logic and, if not, just how far through the hierarchy of intermediate (higher-order) logics does the resulting independence of connectives and quantifiers persist?

If one restricts attention to a reasonably standard list of primitives, namely $\neg$, $\vee$, &, $\rightarrow$, $\exists$, and $\forall$, then a fairly pleasant pattern emerges.[1] Starting

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