Notre Dame Journal of Formal Logic
Volume 26, Number 4, October 1985

Classification Theory Over a Predicate I

ANAND PILLAY and SAHARON SHELAH

Introduction In this paper, the scene is set for the study of classification over a predicate. Let T be a complete first-order theory with among other things a unary predicate P. Instead of studying the structure and number of models of T we are now interested in the structure and number of models M of T *over* M^P (where M^P is the substructure of M with universe P^M). So for example we let $I_T(\lambda, \mu)$ be the greatest κ such that there are N of power μ and κ models $M \models T$, with $M^P = N$ and $|M| = \lambda$, which are pairwise non-N-isomorphic.

In Section 2 it is pointed out that, given N, those $M \models T$ with $M^P = N$ can be to some extent coded by L_2-reducts of *expansions* of N to T^* where $L(T^*) \supseteq L_2 \supseteq L(T)$, for suitable L_2, T^*. So in Section 1 the following is examined: given $L_1 \subseteq L_2 \subseteq L_3$ and T^* a theory in L_3, what are the possible numbers of expansions of N to L_2-reducts of models of T^* as N ranges over L_1-structures? This generalizes the context of the Chang-Makkai theorem (see [1]) and results in [6]. Some finer results are also obtained.

Such results are used to show that if $I_T(\lambda, \lambda)$ is not too big then for every $M \models T$, $\bar{a} \in M$, $tp(\bar{a}/P^M)$ is definable.

In Section 3 some stability-type notions are introduced. The general context here is: given $M \models T\, A \subseteq M$, $A \supseteq P^M$ one should study the space of those types p over A which can be realized in some $N \models T$ with $P^N = P^A$ $(= P^M)$. In future work fairly complete answers to spectrum problems (e.g., analogues of Morley's theorem) will be given by studying the space of such types for successively more "complicated" such A, using the techniques similar to [9]. Here we essentially consider only the case $A = M \models T$ and prove some non-structure theorems.[1]

1 Let us first establish notation for this section. $L_1 \subseteq L_2 \subseteq L_3$ are first-order languages (with equality), $L_2 - L_1 = \{P_i\colon i < \kappa\}$, and T is a theory in L_3 such that $T \models \exists x \exists y\, (x \neq y)$.

*Partially supported by NSF Grant-DMS 84-01713.

Received December 14, 1979, revised September 1984