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1-Consistency and the Diamond

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1 Introduction It is well known that the set of (Gödel numbers of) sentences of arithmetic that are consistent with classical first-order arithmetic with induction (Peano Arithmetic (PA)) is Π-1 complete. Solovay showed in [5] that the propositional modal logic characterizing consistency in PA is the system of modal logic known variously as G, GL, and L: the formulas of modal logic that are theorems of G are precisely those that are provable in PA under all substitutions of sentences of arithmetic for atoms p_0, p_1, p_2... of modal logic, the diamond $\Diamond$ and box $\Box$ of modal logic being respectively interpreted as the consistency predicate $Con(x)$ and the provability predicate $Bew(x)$ of arithmetic.[1] In [3], building upon Solovay's work, I showed that the set of sentences that are ω-consistent with PA is Π-3 complete and that the system G is also the modal logic characterizing ω-consistency. Thus despite the greater complexity of its definition, there is a natural and easily definable class of properties in respect of which ω-consistency does not differ from (simple) consistency.

A theory T in the language of arithmetic is said to be *1-inconsistent* if for some *primitive recursive* formula Rx, T implies $\exists x{-}Rx$ and also implies $R\mathbf{n}$, for every natural number n; T is 1-consistent if it is not 1-inconsistent. The definition of ω-consistency thus differs from that of 1-consistency only in lacking the qualifier "primitive recursive". Obviously, every ω-consistent theory is 1-consistent and every 1-consistent theory is consistent; neither converse holds.

1-consistency was first defined by Kreisel. Some interesting facts about it are: (i) A modification of the finite version of Ramsey's theorem, due to Paris and Harrington, turns out to be equivalent in PA to the assertion of 1-consistency, as do a number of other "mathematically interesting, non-self-referential" undecidable sentences devised by other authors. (ii) In his proof of the incompleteness theorems, Gödel constructed a sentence which he showed to be undecidable in the system under consideration on the assumption that the system is ω-consistent. As Kreisel observed, however, this assumption is unnecessarily strong; the assumption that the system is 1-consistent suffices to show that the sentence Gödel constructed is undecidable. (Rosser showed that a certain

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