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A Note on the Hanf Number of Second-Order Logic

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The Hanf number of a logic L is the least cardinal κ such that every sentence of L that has a model of power at least κ has arbitrarily large models. The Hanf number κ^{Π} of second-order logic is very large; for example, it is readily seen to exceed the first measurable cardinal (if there is one). In fact, Barwise [1] showed that one cannot prove that κ^{Π} exists within the theory ZF_1, which is ZFC with the full subset schema but with collection only for $\Sigma_1(\mathcal{P})$ formulas. (Here $\mathcal{P}$ is a unary function symbol, and the power set axiom reads: $\forall x \forall y(y \subseteq x \leftrightarrow y \in \mathcal{P}(x))$.) Moreover, within ZF_1 he proved that $R_{\kappa^{\Pi}} \models ZF_1$, and in fact κ^{Π} is the $(\kappa^{\Pi})^{\text{th}}$ cardinal with this property. Friedman [3] improved this result by showing that even in the weaker theory $T = ZF_0 + (\beta)$, where $ZF_0 = KP(\mathcal{P}) +$ [Power set axiom], if κ^{Π} exists then $R_{\kappa^{\Pi}} \prec_{\Sigma_1(\mathcal{P})} V$.[1]

In this short note we use Friedman's result to give a new characterization of κ^{Π} (Theorem 1 below). A related characterization is given in Väänänen [5] (Corollary 5.7):

(1) $\kappa^{\Pi} = sup\{\alpha\colon \alpha \textit{ is } \Sigma_2\textit{-definable}\}$,

where a set S is Σ_2-definable if the predicate "$x \in S$" is a Σ_2-definable predicate of set theory.[2] (Väänänen's result is actually more general.) Here is an outline of a proof of (1). For $\geq$, if $\phi(x)$ is a Σ_2 (or $\Sigma_1(\mathcal{P})$) definition of "$x \in \alpha$" then consider the following sentence ψ of second-order logic, which holds in $(R_\kappa, \in)$ if κ is least such that $\phi(x)$ defines "$x \in \alpha$" in $(R_\kappa, \in)$:

$$\psi \equiv \text{"The universe is of the form } (R_\delta, \in)\text{"} \wedge (\exists! \beta)(\forall x)(\phi(x) \leftrightarrow x \in \beta) \wedge \forall \beta[\forall x(\phi(x) \leftrightarrow x \in \beta) \rightarrow \forall \gamma \exists x \in \beta(R_\gamma \models \neg\phi(x))] \ .$$

Then ψ has a model of power at least $|\alpha|$, but it's easy to see that ψ does not have arbitrarily large models. For $\leq$, observe that if ϕ is a sentence of second-

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