Notre Dame Journal of Formal Logic
Volume 26, Number 2, April 1985

Some Exact Equiconsistency Results in Set Theory

LEO HARRINGTON and SAHARON SHELAH

Since the invention of forcing there have been innumerable examples of consistency results in set theory. These usually show that *ZFC* + (something) is consistent provided that *ZFC* itself is consistent; hence they may be viewed as equiconsistency results. More recently there have been many forcing arguments that need more than just the consistency of *ZFC*; they assume the consistency of certain large cardinals in addition. Sometimes these are still exact equiconsistency results: for example the negation of Kurepa's hypothesis is equiconsistent with an inaccessible cardinal. Sometimes there is a wide gap between consistency strengths: for example, Magidor's model for the failure of the singular cardinal problem (*SC*) uses somewhat more than a supercompact, while *SC* itself is only known (by work of Mitchell) to imply inner models with many measurable cardinals. And sometimes there is a gap originally, but the gap is eventually closed: for example, Silver's model for Chang's conjecture uses an Erdos cardinal, and Jensen has shown that Chang's conjecture implies the existence of an Erdos cardinal in the core model. In this paper, we present a few more results exemplifying this last possibility. We show:

Theorem A *The following are equiconsistent* (*modulo ZFC of course*):
(i) *the existence of a Mahlo cardinal*
(ii) *every stationary subset of* $\aleph_2$ *consisting just of cofinality* ω *ordinals is stationary in some ordinal* $< \aleph_2$.

Comments: Baumgartner [1] has shown that (ii) is consistent assuming the existence of a weakly compact cardinal. It is known [2] that $\square_{\omega_1}$ implies that (ii) is false, and it is also known [2] that, unless $\aleph_2$ is Mahlo in L, $\square_{\omega_1}$ holds; thus (ii) implies that $\aleph_2$ is Mahlo in L. So the gap in this case is between Mahlo and weakly compact.

Received January 15, 1982; revised September 13, 1984