Notre Dame Journal of Formal Logic
Volume 26, Number 1, January 1985

A Note on Satisfaction Classes

ROMAN KOSSAK

1 Introduction Let L be the language of Peano Arithmetic (PA). If M is a nonstandard model of PA then any arithmetization of L determines a nonstandard language Form (M), consisting of the formulas in the sense of M. Investigation of this language was begun in Robinson's [11]. Satisfaction classes were introduced by Krajewski in [9] in a study of the semantics of Form (M).

It is a common practice in the literature to give only a "rough idea" of satisfaction class and refer for a precise definition to [9]. This sometimes leads to misunderstandings, especially when one has to distinguish full satisfaction classes from those which are not full. For the reader's convenience in what follows, we present Krajewski's definition with some minor changes on which we comment later.

In the definition below, the symbols $\neg$, $\vee$, & denote functions on Form (M) or (Form (M))^2, respectively, given by the arithmetization. This applies also to symbols $\exists v_k$, $\forall v_k$, where v_k is a variable of Form (M).

We say that $\Phi \subseteq \mathrm{Form}(M)$ is *closed under immediate subformulas* if whenever any of the formulas $\neg\phi$, $\exists v_k \phi$, $\forall v_k \phi$ is in Φ, then ϕ is in Φ, and whenever $\phi \vee \psi$ or $\phi \,\&\, \psi$ is in Φ, then so are ϕ and ψ.

Satisfaction classes on M are certain sets of pairs of the form $\langle \phi, a\rangle$, where $\phi \in \mathrm{Form}(M)$ and a is a valuation for ϕ. So a is a sequence of elements of M with domain corresponding to the set of free variables of ϕ. Using an arithmetical coding of finite sequences we treat satisfaction classes as subsets of M.

1.1 Definition If M is a model of PA, a subset S of M is a *satisfaction class* iff

a. every $x \in S$ is of the form $\langle \phi, a\rangle$, where $\phi \in \mathrm{Form}(M)$ and a is a valuation for ϕ
b. the class $\Phi(S) = \{\phi \in \mathrm{Form}(M) : \exists a \langle \phi, a\rangle \in S \vee \forall a$ (a is a valuation for ϕ) $\Rightarrow \langle \neg\phi, a\rangle \in S\}$ is closed under immediate subformulas
c. if $M \models \phi a$ and $\ulcorner \phi \urcorner$ is the Gödel number of ϕ, then $\langle \ulcorner \phi \urcorner, a\rangle \in S$

Received June 17, 1983; revised February 22, 1984