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# Inaccessible Worlds

I. L. HUMBERSTONE

This note presents some considerations on the logic of inaccessibility from the point of view of the Kripke semantics for modal logic. That is, we are interested in the logical properties of the usual language of propositional modal logic (say with $\sim$, $\wedge$, and $\Box$ as primitive) as enriched by a new intensional primitive $\blacksquare$ with the semantical clause:

[$\blacksquare$] For any model $\langle W, R, V\rangle$ and any point $x \in W$, $\langle W, R, V\rangle \models_x \blacksquare\alpha$ iff for all $y \in W$ such that not $Rxy$, $\langle W, R, V\rangle \models_y \alpha$.

The other connectives receive the clauses familiar from the standard definition of truth (at a point in a model) in Kripke semantics.[1] We write $\Diamond(\blacklozenge)$ for the dual of $\Box(\blacksquare)$. In the enriched language, thus understood, many classes of frames are expressible which are not expressible in the customary language. (We speak of a formula's expressing a class of frames when the formula is valid on all and only frames in the class, where a formula is valid on a frame $\langle W, R\rangle$ just in case it is true at every point in every model $\langle W, R, V\rangle$ on that frame.) Here are five simple examples:

1. The class of irreflexive frames, expressed by: $\blacksquare p \rightarrow p$
2. The class of asymmetric frames, expressed by: $p \rightarrow \Box\blacklozenge p$
3. The class of intransitive frames, expressed by: $\blacksquare p \rightarrow \Box\Box p$
4. The class of strongly connected frames (i.e., frames $\langle W, R\rangle$ such that for all $x, y \in W$ either $Rxy$ or $Ryx$), expressed by: $p \rightarrow \blacksquare\Diamond p$
5. The class of universal frames (i.e., frames $\langle W, R\rangle$ with $R = W \times W$), expressed by: $\blacksquare(p \wedge \sim p)$.

Apart from any purely technical interest such examples may have, some of them are of obvious relevance when the unenriched language is thought of in tense-logical terms (thinking of $\Box$ as Prior's $G$, [5]). Further applications are suggested by the fact that we may sometimes wish to consider an operator $O$

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