Notre Dame Journal of Formal Logic
Volume 24, Number 2, April 1983

Compactness via Prime Semilattices

R. H. COWEN

1 Introduction Compactness is certainly one of the most fruitful concepts of general topology. Topologically inspired notions of compactness have also proven useful in logic (see [3], [9]) and measure theory (see [8]). In this paper we introduce a definition of compactness for subsets of a prime semilattice. Prime semilattices were introduced by Balbes [2] and their algebraic structure seems just right for presenting the ideas which underlie many compactness arguments.

2 Prime semilattices and Wallman's lemma Let $\langle S, \leqslant\rangle$ be a partially ordered set and suppose $T \subseteq S$. The greatest lower bound, or meet, of T, if it exists, will be denoted by $\wedge T$. The least upper bound, or join, if it exists, will be denoted by $\vee T$. If T is finite, $T = \{t_1, \ldots, t_n\}$, we shall write $t_1 \wedge \ldots \wedge t_n$ and $t_1 \vee \ldots \vee t_n$ for the meet and join of T, respectively.

A partially ordered set $\langle S, \leqslant\rangle$ is a (meet) *semilattice* if every finite, nonempty, set has a meet. A semilattice is said to be *prime* if whenever $s \in S$ and $s_1 \vee \ldots \vee s_n \in S$ then $(s \wedge s_1) \vee \ldots \vee (s \wedge s_n) \in S$ and $s \wedge (s_1 \vee \ldots \vee s_n) = (s \wedge s_1) \vee \ldots \vee (s \wedge s_n)$.

Let $\langle S, \leqslant\rangle$ be a semilattice and suppose $I \subseteq S$, $I \neq \phi$. I is an *ideal* of the semilattice $\langle S, \leqslant\rangle$ if $s \in I$ and $t \leqslant s$ implies $t \in I$; if, in addition, $s, t \in I$ and $s \vee t \in S$ implies $s \vee t \in I$, I will be called a *regular* ideal.

Suppose I is an ideal of the semilattice $\langle S, \leqslant\rangle$. A subset $W \subset S$ *avoids* I if $\wedge W \notin I$; W *finitely avoids* I if $\wedge W_0 \notin I$, for every finite $W_0 \subset W$. The following theorem generalizes a lemma of Wallman [12].

Theorem 1 *Let $\langle S, \leqslant\rangle$ be a prime semilattice and I a regular ideal of $\langle S, \leqslant\rangle$. Suppose $\{b_j\}_{j\in J}$ is a subcollection of S which finitely avoids I and $b_j = a_{j1} \vee \ldots \vee a_{jn_j}$, $j \in J$. Then there is a function f with domain J such that $\{a_{jf(j)}\}_{j\in J}$ finitely avoids I.*

Received August 17, 1981