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Consistent, Independent, and Distinct Propositions. III: Modalities in S6

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In [3] McKinsey showed that S2 has infinitely many nonequivalent modalities, viz., the modalities $\Diamond p$, $\Diamond\Diamond p$, $\Diamond\Diamond\Diamond p$, ... Prior ([4], p. 125) conjectured that there are infinitely many nonequivalent modalities in S6. The conjecture readily follows from the results of [6] and [7]. S6 is a subsystem of both S10 and S11, and in both S10 and S11 we have the theorems $\sim(\mathbf{P}_s \prec \mathbf{P}_t)(0 \leqslant s < t)$ for $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \ldots$, and hence that none of $\mathbf{P}_s \prec \mathbf{P}_t$ is a theorem. Hence the modalities $\sim\Diamond p$, $\Diamond\Diamond p$, $\Diamond\sim\Diamond\Diamond p, \ldots$ are all nonequivalent in S10 and S11, and hence in S6. In this note we show that the modalities $\Diamond p$, $\Diamond\Diamond p$, $\Diamond\Diamond\Diamond p, \ldots$ are all nonequivalent in S6. Lewis ([2], p. 499) described $\Diamond\Diamond p \prec \Diamond p$ and $\Diamond\Diamond p$ as 'contrary assumptions' in the field of S2. (This is an error.) If $\Diamond\Diamond p \prec \Diamond p$ is assumed clearly all the aforementioned modalities collapse to $\Diamond p$. We note that if we assume $\Diamond\Diamond p$ all of them remain nonequivalent.

Consider the matrix described in [6], pp. 402-403. Our matrix $\mathfrak{M}$ is obtained from it by the following modification: $P\{n\} = \{2, 3, \ldots, n, n + 2\}$ $(n \geqslant 3)$. By Theorem 2 of [6] (p. 402), $\mathfrak{M}$ is a σ-regular S6-matrix. We show that none of the following is a theorem of S6: $\Diamond^s p \prec \Diamond^t p$ $(s > t \geqslant 1)$. We first note that $P^n 0 = \{2, 3, \ldots, 2n - 1, 2n + 1\}$ $(n \geqslant 2)$. We proceed by induction. $P^{n+1}0 = P(P^n 0) = P\{2, 3, \ldots, 2n - 1, 2n + 1\} = P\{2\} \cup [P\{4\} \cup P\{6\} \cup \ldots \cup P\{2n - 2\}] \cup [P\{3\} \cup P\{5\} \cup \ldots \cup P\{2n + 1\}] = P\{2\} \cup P\{2n - 2\} \cup P\{2n + 1\} = \{2, 3\} \cup \{2, 3, \ldots, 2n - 2, 2n\} \cup \{2, 3, \ldots, 2n + 1, 2n + 3\} = \{2, 3, \ldots, 2n + 1, 2n + 3\}$. Now suppose that $\vdash_{S6} \Diamond^s p \prec \Diamond^t p$. Hence, since $\mathfrak{M}$ is an S6–matrix, by Definitions II.17.16 [5], for $x \in M$, $P^s x \Rightarrow P^t x \in D$. By Theorem III.6 [5], $P^s x \leqslant P^t x$. Let $x = 0$. If $t = 1$, $\{2, 3, \ldots, 2s - 1, 2s + 1\} = P^s 0 \leqslant P0 = \{3\}$. If $t > 1$, $\{2, 3, \ldots, 2s - 1, 2s + 1\} = P^s 0 \leqslant P^t 0 = \{2, 3, \ldots, 2t - 1, 2t + 1\}$. By Definition II.10 [5], both are contradictions.

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