Notre Dame Journal of Formal Logic
Volume 29, Number 4, Fall 1988

There are Denumerably Many Ternary Intuitionistic Sheffer Functions

DJORDJE ČUBRIĆ*

In [1] Došen asks what is the number of mutually nonequivalent ternary indigenous Sheffer functions for $\{\rightarrow,\wedge,\vee,\neg\}$ in the intuitionistic propositional calculus (IPC). The answer is: denumerably many.

Following [2] we shall say that a set of functions F is an indigenous Sheffer set for a set of functions G iff every member of G can be defined by a finite number of compositions from the members of F and vice versa. A function f is an indigenous Sheffer function for G iff $\{f\}$ is an indigenous Sheffer set for G. The n-ary propositional functions f_1 and f_2 are mutually equivalent iff for some permutation P of the sequence $A_1,\ldots,A_n$ in the propositional calculus we can prove $f_1(A_1,\ldots,A_n) \leftrightarrow f_2(P)$. We work all the time in IPC. Expressions of the form $\vdash A$ (or $\nvdash A$) mean that A is provable (or unprovable) in IPC.

Kuznetsov [3] and Hendry [2] have shown that there is no binary indigenous Sheffer function for $\{\rightarrow,\wedge,\vee,\neg\}$ in IPC. The first example of a ternary indigenous Sheffer function was given in [3]. Here we use one of the three examples given in [1].

The Rieger–Nishimura Lattice of one variable X, RNL(X) is recursively defined as follows: $P_0(X) = X \wedge \neg X$, $P_1(X) = X$, $P_2(X) = \neg X, P_\infty(X) = X \rightarrow X$, $P_{2n+3}(X) = P_{2n+1}(X) \vee P_{2n+2}(X)$, $P_{2n+4}(X) = P_{2n+3}(X) \rightarrow P_{2n+1}(X)$, for $n \geq 0$. For every $i > j$, $\nvdash P_i(X) \rightarrow P_j(X)$ (see [5] or [4]).

First, we have one simple lemma:

Lemma *For every* $i \geq 5$:
(1) $\vdash \neg X \wedge P_i(X) \leftrightarrow \neg X$
(2) $\vdash P_i(\bot)$.

Proof: (1) *For every* $i \geq 5$, we have $\neg X \vdash P_i(X)$ directly from RNL(X). We obtain (2) by using $\vdash \neg\bot$ and (1).

*I wish to thank Kosta Došen for the help he gave me in writing this note.

Received October 27, 1986; revised February 9, 1987