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Many-Sorted Elementary Equivalence

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Introduction Let us consider a many-sorted language $\mathcal{L}$. Any $\mathcal{L}$-theory $\mathfrak{T}$ can be effectively replaced by an equally powerful $\mathcal{L}^*$-theory $\mathfrak{T}^*$, where $\mathcal{L}^*$ is a one-sorted language canonically associated with $\mathcal{L}$ (most often, $\mathcal{L}^*$ contains a unary predicate S^s for each sort s of $\mathcal{L}$). Such a remark appears in the first paragraphs of many texts dealing with many-sorted theories, e.g. [3], p. 13; [5], ch. 5; or [7], ch. XII.

On the other hand, some many-sorted notions cannot be directly transposed to the corresponding one-sorted notions (see, for example, [4]).

In this paper, we will study how the many-sorted elementary equivalence can be transposed into one-sorted elementary equivalence. More precisely, if $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{L}$-structures we will see when $\mathfrak{M} \equiv \mathfrak{N}$ implies, or is implied by, $\mathfrak{M}^* \equiv^* \mathfrak{N}^*$, where $\mathfrak{M}^*$ and $\mathfrak{N}^*$ are $\mathcal{L}^*$-structures canonically associated with $\mathfrak{M}$ and $\mathfrak{N}$ (in a way which will be made more precise later), and $\equiv$ and $\equiv^*$ denote respectively the $\mathcal{L}$- and $\mathcal{L}^*$-elementary equivalence relations.

First, we will study, as an example, the case where $\mathcal{L}$ is $\mathcal{L}_{TT}$, the language of Simple Type Theory, with four different ways to build $\mathcal{L}^*$. Then we will characterize the many-sorted languages for which the results for $\mathcal{L}_{TT}$ can be generalized.

1 An example: $\mathcal{L}_{TT}$ If $\mathcal{L}$ is $\mathcal{L}_{TT}$, then the set of sorts of $\mathcal{L}$ thus is ω, and its only nonlogical symbol is the binary relational symbol $\in$. Hence, if $\mathfrak{A}$ is an $\mathcal{L}$-structure, then $\mathfrak{A}$ will be of the form

$$\mathfrak{A} = (A^0, A^1, \ldots ; \in_{\mathfrak{A}}),$$

where

$$\in_{\mathfrak{A}} \subset \bigcup_{i \in \omega} A^i \times A^{i+1}. \tag{1}$$

As usual, we will impose that

$$\forall i \in \omega,\ A^i \neq \emptyset,$$

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