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The Lindenbaum Construction and Decidability

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Let L be a set of sentences in a formal language – propositional, predicate, modal, epistemic, etc.[1] It is assumed that the syntax of the language is effective and that it includes the standard sentential connectives. It is also assumed that the language has a deductive system in which the axioms and rules of inference are recursively enumerable and all truth-functional tautologies are theorems.

The set L is said to be *Post-complete* if L is consistent and has no consistent, proper extension. That is, if L is Post-complete then for every sentence Φ, either Φ is in L or the set $L \cup \{\Phi\}$ is not consistent.[2]

Lemma 1 *If a recursively enumerable set of sentences is Post-complete, then it is recursive.*

Proof: Let L be recursively enumerable and Post-complete. To determine if a given sentence Φ is in L, simultaneously enumerate L and enumerate the consequences of $L \cup \{\Phi\}$. If Φ is in L, then eventually Φ will appear in the first enumeration. If Φ is not in L, then, by the Post-completeness of L, a contradiction will eventually appear in the second enumeration.

The Lindenbaum proof of maximal consistency occurs in several contexts in mathematical logic (see, for example, [2], pp. 64–65). In virtually all cases, the technique yields a proof that every consistent set of sentences has a Post-complete extension. It is often noted that the proof is not constructive. This amounts to an observation that the proof does not provide an effective method for enumerating the members of the indicated Post-complete extension, even if an enumeration procedure is available for the given set of sentences.

The Rosser extension of the Gödel incompleteness theorem (see [2], pp. 145–148) indicates that in the case of full predicate logic, there cannot be a general constructive proof of the Lindenbaum theorem. Indeed, the Rosser theorem

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