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A Note on Arbitrarily Complex Recursive Functions

CARL H. SMITH*

1 Introduction The first result that one studies in abstract complexity theory is that there are arbitrarily difficult to compute recursive functions. In this note we prove a best possible generalization of that result. We show that there are arbitrarily sparse {0,1} valued recursive funcions such that any finite variant of the constructed function is arbitrarily complex. Other potential strengthenings are shown to fail. The intricate nature of the main proof is necessitated only by the existence of pathological complexity measures. For the purpose of the main result, pathological complexity measures are shown to be ones where padding cannot be accomplished without significantly altering the complexity of the program being padded.

Rabin ([20]) first proved, in a machine dependent fashion, that there exist almost everywhere arbitrarily complex recursive functions with range {0,1}. His result was generalized to the machine independent case in [2]. Both proofs employ a nonconstructive element. A completely constructive proof of the existence of infinitely often arbitrarily complex recursive functions appears in [10]. The relative difficulty of proving functions almost everywhere arbitrarily complex, as opposed to infinitely often arbitrarily complex, is discussed in [8]. The nonconstructivity of the original Rabin–Blum proof (and other results from abstract complexity theory) motivated Lipton ([15]) to consider restricted models of arithmetic where only constructive proof techniques were allowed. This model was claimed adequate for research in theoretical computer science. However, it was shown in [11] that it was consistent with the model studied by Lipton to believe in some obviously false assertions about the complexity of computations,

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