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Partially Generic Formulas in Arithmetic

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Introduction The following problem arose in connection with a question concerning interpretability (cf. [4]). Let T be a consistent recursively enumerable (r.e.) theory containing a sufficient amount of arithmetic. We do not assume that T is Σ_1^0-*sound*, i.e., that only true Σ_1^0 sentences are provable in T. Next, let S be any r.e. theory, set $\mathrm{Th}_T(S) = \{\phi: \exists q T \vdash \mathrm{Pr}_{[S \restriction q]}(\bar{\phi})\}$, and let $\sigma(x)$ be any formula numerating S in T. Then if $\phi \in \mathrm{Th}_T(S)$, then $T \vdash \mathrm{Pr}_\sigma(\bar{\phi})$. The problem is now if there is a (Σ_1^0) formula $\sigma(x)$ numerating S in T such that $\mathrm{Pr}_\sigma(x)$ numerates $\mathrm{Th}_T(S)$ in T. (This is, of course, true if T is Σ_1^0-sound.) As was shown in [4] (Lemma 2), the answer is affirmative. The proof uses a result of Guaspari ([1]) on partially conservative sentences. The purpose of this note is to describe a fairly general method (fixed-point construction) by means of which a more direct proof can be obtained and to give some examples of applications of this method including the result of Guaspari just mentioned. This paper may be compared with Smoryński's paper ([8]).

1 ***Preliminaries*** Let T be a consistent r.e. theory. For simplicity we shall assume that T is an extension of Peano arithmetic P. (Notation and terminology not explained here are standard.) Let G be a new predicate. Formulas containing G will be written $\zeta(G;\tilde{x})$, $\chi(G;\tilde{x})$. (Here $\tilde{x}$ is short for $x_0,\ldots,x_{r-1}$. Similarly we write $\tilde{k}$ for $k_0,\ldots,k_{r-1}$ and $\bar{\tilde{k}}$ for $\bar{k}_0,\ldots,\bar{k}_{r-1}$.) For simplicity we assume that G is monadic. The extension of the results of Sections 2 and 4 to formulas containing polyadic predicates is perfectly straightforward. If $\xi(x)$ is any formula, then $\zeta(\xi;\tilde{x})$ is obtained from ζ by replacing G by $\xi(x)$ avoiding clashes of variables in the usual way. To prevent confusion we sometimes use the notation $\lambda x\xi(x)$. In the following we always assume that χ is *positive* in G in the sense that for any arithmetical formulas $\xi_0(x)$ and $\xi_1(x)$,

$$P \vdash \chi(\xi_0;\tilde{x}) \wedge \forall x(\xi_0(x) \to \xi_1(x)) \to \chi(\xi_1;\tilde{x}).$$

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