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Reflections on Church's Thesis

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Over fifty years after I first heard Church propose his thesis, about which I have meanwhile often written, can I find anything more to say concerning it? I have been introduced to much of the recent literature in which Church's thesis is discussed by the excellent scholarly volume [29] of Judson Webb.[1] Its bibliography, which of course covers many topics besides Church's thesis, includes over 300 items, about half of them published since 1960. It is nevertheless not quite complete; thus Post [25], Markov [21] and [22], and Smullyan [26] are not listed, although they are devoted to expounding some of the newer equivalent versions of Church's thesis. Also, a new book from the Russian school has just appeared: Markov (posthumous) and Nagornyi [23].

It is a recurrent theme in Webb [29] that Gödel's (first) incompleteness theorem of [8] gave "protection" to Church's thesis; thus, if, contrary to the incompleteness theorem, a system F such as Gödel considered were complete (i.e., for each closed formula A, either $\vdash_F$ A or $\vdash_F \neg$A) and gave correct results (say, satisfied Gödel's hypothesis of ω-consistency), then in Kleene's effective enumeration (with repetitions) $\phi_0(x), \phi_1(x), \ldots, \phi_z(x), \ldots$ (where $\phi_z(x) = U(\mu y T_1(z,x,y))$ of all the 1-place partial recursive functions (including all the 1-place general recursive functions), we could effectively complete the definitions of all the functions which are not total (leaving those that are total unchanged) getting $\bar{\phi}_0(x), \bar{\phi}_1(x), \ldots, \bar{\phi}_z(x), \ldots$, by putting

$$\bar{\phi}_z(x) = \begin{cases} U(y) & \text{if } T_1(z,x,y), \\ 0 & \text{if } \vdash_F \forall \mathrm{y} \neg \mathrm{T}_1(\mathbf{z},\mathbf{x},\mathrm{y}). \end{cases}$$

That is, for given z and x, we search effectively through the numbers $y = 0,1,2,\ldots$ for the first one such that *either* $T_1(z,x,y)$ holds (on finding which we put $\bar{\phi}_z(x) = U(y)$) or y is the Gödel number of a proof in F of $\forall \mathrm{y} \neg \mathrm{T}_1(\mathbf{z},\mathbf{x},\mathrm{y})$ (on finding which we put $\bar{\phi}_z(x) = 0$).[2] Now by diagonalizing we would get $\bar{\phi}_x(x) + 1$ as an effective total 1-place function which is not general recursive,[3] contradicting Church's thesis. So, as Webb correctly stresses, if we hadn't the

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