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On n-Equivalence of Binary Trees

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Summary and introduction This note presents a simple characterization of the class of all trees which are n-elementary equivalent with B_m: the binary tree with one root all of whose branches have length m (for each pair of positive integers n and m). Section 1 contains some preliminaries. Section 2 introduces the class $Q(n)$ of binary trees and proves that every tree in it is n-equivalent with B_m whenever $m \geq 2^n - 1$. Section 3 shows that, conversely, each n-equivalent of a B_m with $m \geq 2^n - 1$ belongs to $Q(n)$. Finally, all n-equivalents of B_m for $m < 2^n - 1$ are isomorphic to B_m.

1 Preliminaries Define the relation $\equiv^n$ between models of the same finite vocabulary (not containing function-symbols) using induction on n by

(1) $A \equiv^0 B$ iff A and B have the same true atomic sentences
(2) $A \equiv^{n+1} B$ iff both
  (i) $\forall a \in A \exists b \in B (A, a) \equiv^n (B, b)$
  (ii) $\forall b \in B \exists a \in A (A, a) \equiv^n (B, b)$.

Also, when $\underline{a} \in A^k$, define the first-order (!) formula $\sigma_{\underline{a}}^n(x_0, \ldots, x_{k-1})$ of quantifier rank n by

(1′) $\sigma_{\underline{a}}^0$ is the conjunction of all formulas with at most $x_0, \ldots, x_{k-1}$ free satisfied by $\underline{a}$ in A which are either atomic or negated atomic
(2′) $\sigma_{\underline{a}}^{n+1}$ is $\forall x_k \bigvee_{b \in A} \sigma_{\underline{a}^\frown\langle b\rangle}^n \wedge \bigwedge_{b \in A} \exists x_k \sigma_{\underline{a}^\frown\langle b\rangle}^n$.

For a definition of the Ehrenfeucht-game and a proof of the next lemma (be it in the context of linear orderings) I refer to [1], pp. 93–96, 247–252 and 359–361.

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