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Survey of Generalizations of Urquhart Semantics

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0 Introduction It has often been felt that the Intuitionist account of implication is too inclusive, the most thoroughly argued case against it being that of Anderson and Belnap. In my opinion the problem is not the irrelevance of antecedents but the relationship between them, which Intuitionism takes to be conjunction. Thus it has

$$((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$$
$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C),$$

which are equivalent[1] to $A \rightarrow (B \rightarrow (A \wedge B))$ and $A \rightarrow (B \rightarrow A)$, and to $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$, respectively. Consequences of the second formula, which I consider to be counterintuitive, include $(A \wedge (A \rightarrow B)) \rightarrow B$, $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$, and, together with the contrapositive, $(A \rightarrow B) \rightarrow (\bar{A} \vee B)$ and $(A \rightarrow \bar{A}) \rightarrow \bar{A}$. If the relationship between antecedents is not conjunction, one can introduce some other symbol for it, say $\circ$, referred to as fusion or intensional conjunction. In view of these remarks it is appropriate to consider the logic often called $R^+ - W^2$, with axiom schemas

$\vdash A \rightarrow A$
$\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$
$\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
$\vdash (A \wedge B) \rightarrow A$ $\vdash (A \wedge B) \rightarrow B$
$\vdash ((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
$\vdash A \rightarrow (A \vee B)$ $\vdash (B \rightarrow (A \vee B)$
$\vdash ((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
$\vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
$\vdash A \rightarrow (B \rightarrow (A \circ B))$
$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C)$

and the rules

if $\vdash A$ and $\vdash B$ then $\vdash A \wedge B$
if $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$.

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