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The Modal Logic of 'All and Only'

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1 The modal logics of 'All' and 'Only' We work with the customary language of modal propositional logic, in which formulas are built in the usual way by application of some functionally complete set of truth-functional primitive connectives alongside the singulary connective '$\Box$', from a stock of sentence letters (or 'propositional variables'), of which there are taken to be countably many. For further background and terminology not explained here, see [6]. This language is interpreted by means of frames $\langle W, R\rangle$ and models $\langle W, R, V\rangle$ thereon, with various options being open for the definition of truth of a formula A at a point x in such a model, notated '$\mathfrak{M} \models_x A$' (where $\mathfrak{M} = \langle W, R, V\rangle$ and $x \in W$). We consider only variations on the clause governing $\Box$-formulas in the otherwise standard inductive definition of the $\models$-relation. The contrast between the following pair of clauses, of which the first figures in the standard definition, is quite interesting:

[All] $\mathfrak{M} \models_x \Box A$ iff for all $y \in W$, if xRy then $\mathfrak{M} \models_y A$
[Only] $\mathfrak{M} \models_x \Box A$ iff for all $y \in W$, if $\mathfrak{M} \models_y A$ then xRy

The weakest logic on the 'all' semantics—the system, that is, which is determined by the class of all frames when truth at a point in a model is as dictated by [All]—is of course the system K, while the logic occupying a similarly minimal position when the 'only' semantics is in force is the system of Karmo in [4], called Anti-K. Recapitulating the details relevant to our present purposes, we recall that K may be axiomatized by closing the class of substitution instances of nonmodal tautologies under modus ponens and the rule:

[K] $$\frac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box B},$$

while for Anti-K this rule is replaced by:

[Anti-K] $$\frac{A \rightarrow (B_1 \vee \ldots \vee B_n)}{(\Box B_1 \wedge \ldots \wedge \Box B_n) \rightarrow \Box A}.$$

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