Notre Dame Journal of Formal Logic
Volume 28, Number 1, January 1987

On the Consistency of the First-Order Portion of Frege's Logical System

TERENCE PARSONS*

It is well known that Frege's logical system of his *Grundgesetze der Arithmetik* [1] is inconsistent. However, Peter Schroeder-Heister (in [3]) has speculated that the first-order portion of this system is consistent. On the surface, this is a somewhat surprising conjecture, because Frege's so-called "abstraction" principle is included in the first-order part of his system, and the somewhat similar abstraction principle of (first-order) naive set theory leads quickly to inconsistency. But Frege's abstraction principle is a prima facie weaker principle. Instead of assuming that any formula determines a set which satisfies that formula, it holds only that coextensive formulas must determine the same "courses of values". That is, instead of this:

Set Theorem $(\exists x)(y)(y \in x \equiv A)$, *for any A not containing x*,

we assume (roughly) this:

Frege $(x)(A \equiv B) \equiv \grave{x}A = \grave{x}B$, *for any A, B.*[1]

It is well known that if quantification over functions is admitted into Frege's system (as Frege himself did) then it is possible to define an analogue of set membership, and the abstraction principle of naive set theory can be shown to follow from Frege's abstraction principle.[2] Russell's paradox quickly follows. But it is not obvious how to do this within the first-order portion of Frege's system. The first goal of this paper is to show that this cannot be done. Schroeder-Heister's conjecture is correct: the first-order portion of Frege's system is consistent.

The second goal of this paper is to explore the significance of the model-construction technique sketched herein for Frege's claims about the arbitrariness of the identification of truth-values with courses of values. Although Schroeder-Heister has shown that Frege's claims on this topic are false, there are some closely related claims that are true and interesting.

*Work on this paper was stimulated by [3], and many of the ideas and techniques used here originated there.

Received November 2, 1984