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Identities and Indiscernibility

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1 Introduction Ehrenfeucht and Mostowski introduced in [4] the method of indiscernibles to show that every consistent theory which has an infinite model has models with arbitrarily large automorphism groups. This method provides a powerful tool for the construction of models with predetermined properties. Morley in his proof of Łoś's conjecture used the notion of a set indiscernibles. By the work of Morley, Shelah, and others this notion became important in characterizing stable theories.

The above notions of indiscernibility are all special cases of a more general one. We define it in Section 2. In solving problems concerning indiscernibility it quite often turns out that the argument is essentially combinatorial in nature and only uses properties of the equivalence relation $\sim$, defined by: $x \sim y$ iff x and y satisfy the same formulas. This gives rise to the definition of so-called identities: pairs (A, E) where E is a certain equivalence relation on $\bigcup_{n\in\omega} A^n$. The first one who fruitfully used identities in model-theoretic problems was Shelah. He proved a compactness theorem for pairs of cardinals [8] and gave a combinatorial proof of Vaught's two cardinal theorem [6]. Independently of Shelah, Benda in [1] introduced identities under the name modeloids as objects worthy of study in their own right.

In Sections 2 and 3 we investigate the notion of identity and we introduce a special class of identities, the complete and homogeneous identities. In Sections 4 and 5 we apply our results about identities to investigate the hierarchy of indiscernibility, e.g., we prove that there exists a collection of complete theories $\{T_X | X \subseteq \omega\}$ such that $T_X \leq T_Y$ iff $X \subseteq Y$.

In Section 5 we show by means of an example that the hierarchy of indiscernibility contains infinite chains and infinite antichains. Also here the advantage of using identities becomes apparent. We adopt all modeltheoretic notations from Chang and Keisler [3]. We specially mention the following: $\bar{a}$, $\bar{b}$, ... denote finite sequences. The length of a sequence $\bar{a}$ is $l(\bar{a})$ and if π is a permutation of $\{0, \ldots, l(\bar{a}) - 1\}$, then $\pi\bar{a}$ is the sequence $\langle a_{\pi(0)}, \ldots, a_{\pi(l(\bar{a})-1)}\rangle$.

If $\bar{a}$ is a sequence and α is a strictly increasing sequence $i_0 < i_1 < i_2 < \ldots < i_{k-1} < l(\bar{a})$, then $\bar{a} \upharpoonright \alpha$ is the sequence:

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