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Sums of Finitely Many Ordinals of Various Kinds

MARTIN M. ZUCKERMAN

Abstract The ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be *pairwise-noncommutative* if for all $i, j = 1, 2, \ldots, n$, if $i \neq j$, then $\alpha_i + \alpha_j \neq \alpha_j + \alpha_i$. For positive integers n and k, let Σ_n be the symmetric group on n letters and let E_n (respectively L_n, S_n, T_n, or P_n) be the set of all k for which there exist n (not necessarily distinct) nonzero ordinals (respectively, limit ordinals, successor ordinals, infinite successor ordinals, or pairwise-noncommutative ordinals) such that $\sum_{i=1}^{n} \alpha_{\phi(i)}$ takes on exactly k values as ϕ ranges over Σ_n. Then for all $n \geq 1$, $E_n = L_n = S_n = T_n$; $min\, P_n = n$, and $max\, P_n = max\, E_n$. Furthermore, $P_1 = E_1$, $P_2 = E_2$, $P_3 = E_3 - \{1, 2\}$, and $P_4 = E_4 - \{1, 2, 3, 11\}$.

1 Introduction Addition of ordinal numbers depends upon the order of the summands. For each positive integer n, the maximum number, m_n, of distinct values that can be assumed by a sum of n nonzero ordinal numbers in all $n!$ permutations of the summands has been calculated by Erdös [1] and Wakulicz [3] and [4]. The first few values of m_n are as follows: $m_1 = 1$, $m_2 = 2$, $m_3 = 5$, $m_4 = 13$, $m_5 = 33$, $m_6 = 81$, $m_7 = 193$, $m_8 = 449$; moreover, it is known that $\lim_{n \to \infty} \frac{m_n}{n!} = 0$.

Let n and k be positive integers. Let Σ_n be the symmetric group on n letters. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any n (not necessarily distinct) nonzero ordinals. We will say that $\alpha_1, \alpha_2, \ldots, \alpha_n$ *yield* k sums if $\left\{\sum_{i=1}^{n} \alpha_{\phi(i)} : \phi \in \Sigma_n\right\}$ is a k-element set. Let E_n be the set of all integers k for which there exist n (not necessarily distinct) nonzero ordinals that yield k sums. It is known that $E_n = \{1, 2, 3, \ldots, m_n\}$ for $n = 1, 2, 3, 4, 6, 7$, and 8 ([2], [5], and [6]), that $E_5 = \{1, 2, 3, \ldots, 29\} \cup \{31, 32, 33\}$ ([3]), and that E_n is properly included in $\{1, 2, 3, \ldots, m_n\}$ for all $n \geq 9$ ([7]).

For every ordinal number $\alpha > 0$, let

(1) $\alpha = \omega^{\lambda_1} a_1 + \omega^{\lambda_2} a_2 + \ldots + \omega^{\lambda_r} a_r$

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