On Power of Singular Cardinals

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Abstract   Using elementary methods we find bounds for the function $2^{\aleph_\alpha}$ for $\aleph_\alpha = \alpha$. Using only ZFC without additional assumptions, when e.g., $\aleph_\alpha$ is strong limit of uncountable confinality:

(1) If there is no weakly inaccessible below $\aleph_\alpha$, then there is no such cardinal below $2^{\aleph_\alpha}$.
(2) If $\aleph_\alpha$ is the first cardinal such that $\lambda = \aleph_\lambda$ with $cf\lambda = \aleph_1$, then $2^{\aleph_\alpha} < \kappa$ when κ is the first cardinal such that $\kappa = \aleph_\kappa$ with cofinality $(2^{2^{\aleph_1}})^+$.

We shall also reprove some of Galvin and Hajnal's results. We do not require any knowledge of earlier results on the subject.

Introduction   We shall deal with the following problem: Given a cardinal λ, what are the possible values of 2^λ? More exactly, given $\aleph_\alpha$, our task is to find an ordinal $\alpha(*)$ as small as possible which will satisfy $\aleph_{\alpha(*)} \geq 2^{\aleph_\alpha}$.

Let us write some basic facts concerning the power operation:

(0) $\alpha < \beta \Rightarrow 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$.
(1) For every α $2^{\aleph_\alpha} > \aleph_\alpha$ (Cantor's theorem).