

On Power of Singular Cardinals

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Abstract Using elementary methods we find bounds for the function 2^{\aleph_α} for $\aleph_\alpha = \alpha$. Using only ZFC without additional assumptions, when e.g., \aleph_α is strong limit of uncountable cofinality:

- (1) If there is no weakly inaccessible below \aleph_α , then there is no such cardinal below 2^{\aleph_α} .
- (2) If \aleph_α is the first cardinal such that $\lambda = \aleph_\lambda$ with $cf\lambda = \aleph_1$, then $2^{\aleph_\alpha} < \kappa$ when κ is the first cardinal such that $\kappa = \aleph_\kappa$ with cofinality $(2^{2^{\aleph_1}})^+$.

We shall also reprove some of Galvin and Hajnal's results. We do not require any knowledge of earlier results on the subject.

Introduction We shall deal with the following problem: Given a cardinal λ , what are the possible values of 2^λ ? More exactly, given \aleph_α , our task is to find an ordinal $\alpha(*)$ as small as possible which will satisfy $\aleph_{\alpha(*)} \geq 2^{\aleph_\alpha}$.

Let us write some basic facts concerning the power operation:

- (0) $\alpha < \beta \Rightarrow 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$.
- (1) For every α $2^{\aleph_\alpha} > \aleph_\alpha$ (Cantor's theorem).

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So only Sections 7, 8, and the last part of the Introduction were written by the author: the author gratefully thanks Avraham and Grossberg for their help.