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The Hanf Number of Stationary Logic

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The Hanf number of a logic is the least cardinal κ such that every sentence with a model of power $\geq\kappa$ in fact has arbitrarily large models. For $L(Q_1)$ where Q_1 = "there exist uncountably many", this number is known to be $\beth_\omega$. For the logic $L(aa)$ or "stationary logic" studied in [1], we show that the Hanf number κ is much larger than $\beth_\omega$, which answers a question from [1].[1] In fact, $\omega < \kappa = \beth_\kappa$ (Theorem 2.5) and κ is at least as large as the Hanf number for logic with quantification over countable sets (Theorem 4.1). In the universe L we show that the Hanf numbers of $L(aa)$ and of second-order logic are the same. The result also holds in $L(D)$ where D is a normal ultrafilter on some cardinal (see Theorems 4.3 and 4.6). We also relate the Hanf number of $L(aa)$ to large cardinals, by giving the consistency (relative to the existence of certain large cardinals) of the assertion that it exceeds many measurable cardinals (Theorem 3.5). The main tool for most of these results is Theorem 3.3, which says that $\aleph_1$ is an anomaly in the following sense: while $L(aa)$ behaves nicely on models of power $\aleph_1$ (as shown in [1], [8]), i.e., there is compactness and completeness, there is nevertheless a sentence ψ of $L(aa)$ with arbitrarily large models, such that ψ describes models of set theory in which every countable subset of the model is an element of the model, as long as the model has power greater than $\aleph_1$.

The groundwork is laid in Sections 1 and 2 by proving that one can in a sense force certain κ-like orderings to appear in all sufficiently large models of a fixed sentence of $L(aa)$. Section 3 contains the main result showing that a sentence ψ exists as described above. We conclude in Section 4 with a number of results on the relationships among the Hanf numbers of $L(aa)$, of logic with a well-ordering quantifier, of logic with a quantifier over countable sets, and of second-order logic. Further such results will appear in [9].

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