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Some Failures of Interpolation in Modal Logic

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Offered here is a relatively simple test for failure of interpolation in propositional modal logic. Additional, and quite important, negative results of this same general sort have recently found their way into the literature. Fine [2] has shown that interpolation fails in quantified *S5* and in a broad class of quantified modal logics when strengthened with the Barcan formula and its converse. And on the propositional level, Maksimova [5], [6] has identified twenty-four normal extensions of *S4* in which interpolation holds but has proven that there exist only thirteen other possible candidates from within that class of logics. The present result, though partially overlapping that of Maksimova, takes us beyond the extensions of *S4* and covers nonnormal logics as well.

The key is found in Lemmon's classical characterization of Halldén-incomplete logics (see [3]).

Theorem 1 *L is Halldén-incomplete if and only if L is the intersection of two logics neither of which contains the other.*

The proof proceeds by showing, in effect, that if α and β share no variables, $\alpha \vee \beta \in L$, $\alpha \notin L$ and $\beta \notin L$, then α and β are theorems of consistent extensions of L. But this fact also yields

Theorem 2 *If L has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in L.*

Proof: Suppose L has only one Post-complete extension C and that $\alpha \vee \beta \in L$, $\alpha \notin L$ and $\beta \notin L$ for some α, β having no variables in common. If p is any variable foreign to α and β, then $\sim\alpha \wedge (p \to p)$ and $\beta \wedge (p \to p)$ have a single variable in common and

$$\sim\alpha \wedge (p \to p) \to \beta \wedge (p \to p) \in L.$$

Assume for *reductio* that $\sim\alpha \wedge (p \to p) \to \gamma$, $\gamma \to \beta \wedge (p \to p) \in L$ for some γ containing only the variable p. Then $\alpha \vee \gamma$, $\sim\gamma \vee \beta \in L$, $\gamma \notin L$ and $\sim\gamma \notin L$, where α and γ share no variables, nor do $\sim\gamma$ and β. It follows that γ and $\sim\gamma$

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