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Iterated Images on Manifolds

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Many results of classical recursion theory carry over nicely when generalized to recursive manifolds. However, this paper shows that a standard classical property of the basic operation of iteration does not hold in the generalized setting. Specifically, the result

(*) If f is p.r., then $\{f^n(x_0)\}_{n=0}^{\infty}$ is r.e. for any x_0

does not carry over to recursive manifolds.

By way of comparison and contrast, consider the state of affairs for uniformly reflexive structures (URS). The result of this paper has a parallel in Friedman's example [1] of a URS with a nonsemicomputable splinter.[1] However, if a URS has one nonsemicomputable splinter, then it cannot, in fact, have any infinite semicomputable splinter; this implication does not transfer to manifolds, as can easily be seen.

To keep the paper self-contained, this paragraph briefly reviews the relevant terminology from [2]. A simple example of a manifold is N^2, written as the disjoint union $\bigcup_{i=0}^{\infty} A_i$, where $A_i = \{i\} \times N$ is enumerated by α_i, with $\alpha_i(j) = (i, j)$. B, a subset of N^2, is *$\mathfrak{A}$-r.e.* iff $\alpha_i^{-1}(B)$ is r.e. for every i. A function f from N^2 to N^2 is *$\mathfrak{A}$-$\mathfrak{A}$-rec* iff for each m and n there exists partial recursive $f_{m,n} : \alpha_m^{-1}(f^{-1}(A_n)) \to N$ such that $f \circ \alpha_m = \alpha_n \circ f_{m,n}$. A *compact* f from N^2 to N^2 is one such that each $f(A_i)$ is contained in a finite union of A_k's. If f is 1-1 and $\mathfrak{A}$-$\mathfrak{A}$-rec such that both f and f^{-1} are compact, f is an *embedding*. And, as usual, for $S \subseteq N$, χ_S denotes the characteristic function of S.

We will show that (*) does not generalize to N^2 (much less to other, more complicated manifolds). In fact, there is an embedding f from N^2 onto N^2 and $x_0 \in N^2$ such that not only is $\{f^n(x_0)\}_{n=0}^{\infty}$ not $\mathfrak{A}$-r.e., but $\alpha_k^{-1}(\{f^n(x_0)\}_{n=0}^{\infty})$ is not r.e. for *any* k.

Proof: Define $h : N^2 \to N^2$ by

$$h(\alpha_k(2n + 1)) = \alpha_{k+1}(2n + 1)$$

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