A Linearly Ordered Topological Space
that is Not Normal

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An old question of Birkhoff asks whether it can be shown that every linearly ordered topological space is normal, without using the axiom of choice ([1], p. 252, and [5]). An example shows that it is not possible.

Let ZF be Zermelo-Fraenkel set theory without the axiom of choice; we assume that ZF is consistent. Let $\mathfrak{D}$ be the *ordering principle*, the axiom which asserts that every set can be linearly ordered ([2], p. 19, and [4], p. 31). Let $\mathfrak{S}$ be the *selection principle*, which says that for every family $\mathfrak{F}$ of sets, each containing at least two elements, there is a function f such that for $X \in \mathfrak{F}$, $f(X)$ is a nonempty proper subset of X ([2], p. 53, and [6], p. 207). It is known that there is a model of ZF and $\mathfrak{D}$ in which $\mathfrak{S}$ is false ([2], p. 95, exercise 11). Assume that $\mathfrak{F} = \{X_\alpha : \alpha \in \Omega\}$ is a family of sets that is a counterexample for $\mathfrak{S}$ in a model of ZF in which $\mathfrak{D}$ is true. We use $\mathfrak{F}$ to form a linearly ordered set such that with its order topology it is not normal. For simplicity we assume that the members of $\mathfrak{F}$ are pairwise disjoint, otherwise we could use a family obtained from $\mathfrak{F}$ by replacing each X_α with the Cartesian product $X_\alpha \times \{\alpha\}$. Let ω be the set of nonnegative integers, let ω^* be the set of negative integers, and let $L = \bigcup_{\alpha \in \Omega} (\omega^* \times \{\alpha\} \cup X_\alpha \cup \omega \times \{\alpha\})$. The construction of L does not require the axiom of choice (see [3], Chap. one, and particularly the bottom of p. 15). Using the principle $\mathfrak{D}$ we assume Ω is linearly ordered and for each $\alpha \in \Omega$, X_α is linearly ordered. For $\alpha \in \Omega$ let $\omega^* \times \{\alpha\}$ and $\omega \times \{\alpha\}$ be ordered according to the usual orders of ω^* and ω. Finally we extend the orders of these subsets of L to a linear ordering of L so that for each $\alpha \in \Omega$ the subset $\omega^* \times \{\alpha\} \cup X_\alpha \cup \omega \times \{\alpha\}$ is ordered with the elements of $\omega^* \times \{\alpha\}$ coming just before those in X_α and the elements of $\omega \times \{\alpha\}$ coming just after those in X_α. Between elements of such subsets corresponding to different members of Ω we order

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