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On the Nonaxiomatizability of Some Logics by Finitely Many Schemas

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Introduction First a suitable definition of an axiom schema is given. Then it is proved that the class of $(\beth_\omega, \aleph_0)$ two-cardinal models, although recursively axiomatizable [12], cannot be axiomatized by finitely many schemata. Essentially the same proof shows that logic with the quantifier "there exist at least κ many", where κ is a strong limit cardinal, cannot be so axiomatized. Conclusions related to the literature are then drawn.

1 Since the publication by Mostowski of [8] in 1957, researchers have expended much effort in the study of logics extending first-order logic. The completeness problem for such logics, naturally enough, has been perhaps the major concern, and has been settled for various logics with varying degrees of success.

At one end of this spectrum lies the work on logic with the quantifier "there exist uncountably many," $L(Q_{\aleph_1})$. Early research by Vaught [13] and Fuhrken [3] in 1964 revealed, respectively, that $L(Q_{\aleph_1})$ is recursively axiomatizable, and that it is countably compact. The techniques used to prove these results were indirect, however, and consequently Vaught's work gave no clue as to what a complete set of axioms for $L(Q_{\aleph_1})$ might be. This shortcoming was remedied spectacularly by the work of Keisler [6] in 1970, wherein he proved that a simple finite collection of schemata sufficed to axiomatize $L(Q_{\aleph_1})$. Moreover, his direct methods yielded important model-theoretic tools for the study of $L(Q_{\aleph_1})$.

At the other end of this spectrum are various "abstract" completeness theorems. These are results which, as in Vaught's work, [13], on $L(Q_{\aleph_1})$, establish that a logic has a recursive set of axioms by indirect means and do not

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