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A Note on Some Weak Forms of the Axiom of Choice

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Abstract Erdös and Tarski proved that in ZFC, if $(P, \leq)$ is a quasi-order that has antichains of cardinality θ for all $\theta < \kappa$, and if κ is singular or $\kappa = \aleph_0$, then $(P, \leq)$ has an antichain of cardinality κ. Some variations of this result are developed as weak forms of the Axiom of Choice.

This note contains some variations of a result of Erdös and Tarski [1] which are developed as weak forms of the Axiom of Choice (AC).

Definition Let $(P, \leq)$ be a quasi-order (i.e., $\leq$ is reflexive and transitive).

Two elements x, y of P are said to be *incompatible* if there does not exist $z \in P$ such that $z \leq x$ and $z \leq y$ (otherwise x and y are said to be *compatible*). A subset I of P is said to be an *antichain* if any two elements of I are incompatible.

A partial order $(P, \leq)$ is a *tree* iff for all $x \in P$, $\{y \in P : y \leq x\}$ is well-ordered by $\leq$. If $(P, \leq)$ is a tree and $x \in P$, then the *height* of x ($ht(x)$) is the order type of $\{y \in P : y \leq x\}$. For each ordinal α, the αth level of P ($lev_\alpha(P)$) is $\{x \in P : ht(x) = \alpha\}$. The *height* of P is the least α such that the αth level of P is empty. A *branch* of P is a maximal chain. Henceforth it will be assumed that all trees are single-rooted (that is, $|lev_0(P)| = 1$).

If $(P, \geq)$ is an upside-down tree then a *strong antichain* is an antichain that has at most one element from each level of P.

$SH(\mu)$ is the hypothesis that no μ-Souslin tree exists.

Erdös and Tarski [1] proved that in ZFC, if $(P, \leq)$ is a quasi-order that has antichains of cardinality θ for all $\theta < \kappa$, and if κ is singular or $\kappa = \aleph_0$, then $(P, \leq)$ has an antichain of cardinality κ. The question of to what extent converses of the result of Erdös and Tarski can be obtained will be somewhat considered. That is, in ZF, is the statement "if $(P, \leq)$ has antichains of cardinality θ for all

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