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The Hanf Numbers of Stationary Logic II: Comparison with Other Logics

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Abstract We show the ordering of the Hanf number of $\mathcal{L}_{\omega,\omega}(wo)$ (well ordering) $\mathcal{L}^c_{\omega,\omega}$ (quantification on countable sets), $\mathcal{L}_{\omega,\omega}(aa)$ (stationary logic), and second-order logic has no more restraints provable in ZFC than previously known (those independence proofs assume *CON*(ZFC only). We also get results on corresponding logics for $\mathcal{L}_{\lambda,\mu}$.

0 Introduction The stationary logic, denoted by $\mathcal{L}(aa)$ was introduced by Shelah [8]. Barwise, Kaufman, and Makkai [1] make a comprehensive research on it, proving for it the parallel of the good properties of $\mathcal{L}(Q)$. There has been much interest in this logic, being both manageable and strong (see Kaufman [5] and Shelah [10]).

Later some properties indicating its affinity to second-order logic were discovered. It is easy to see that countable cofinality logic is a sublogic of $\mathcal{L}(aa)$. By [10], for pairs φ, ψ of formulas in $\mathcal{L}_{\omega,\omega}(Q^{cf}_{\aleph_0})$, satisfying $\vdash \varphi \rightarrow \psi$ there is an interpolant in $\mathcal{L}(aa)$. By Kaufman and Shelah [6], for models of power $> \aleph_1$, we can express in $\mathcal{L}_{\omega,\omega}(aa)$ quantification on countable sets. Our main conclusion is (on the logics see Definition 1.1 or the abstract, on h, the Hanf numbers, see Definition 1.2):

Theorem 0.1 *The only restriction on the Hanf numbers of* $\mathcal{L}_{\omega,\omega}(wo)$, $\mathcal{L}^c_{\omega,\omega}$, $\mathcal{L}_{\omega,\omega}(aa)$, $\mathcal{L}^{II}_{\omega,\omega}$ *are*:
(a) $h(\mathcal{L}_{\omega,\omega}(wo)) \leq h(\mathcal{L}^c_{\omega,\omega}) \leq h(\mathcal{L}_{\omega,\omega}(aa)) \leq h(\mathcal{L}^{II}_{\omega,\omega})$
(b) $h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.

Proof: See 2.1 (neccessity), 2.2, 2.4, 2.5, and 3.3 (all five possibilities are consistent).

The independence results are proved assuming *CON*(ZFC) only and the re-

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